Lab 7

2 a) $y = \sqrt{\frac{1}{t(t+1)}} = \left(\frac{1}{t(t+1)}\right)^{1/2}$

$\Rightarrow \ln y = \ln\left(\frac{1}{t(t+1)}\right)^{1/2} = -\frac{1}{2}\ln t - \frac{1}{2}\ln(t+1)$

$\frac{1}{y}\frac{dy}{dt} = \frac{-1}{2t} - \frac{1}{2(t+1)}$

$\Rightarrow \frac{dy}{dt} = \sqrt{\frac{1}{t(t+1)}}\left(\frac{-1}{2t} - \frac{1}{2(t+1)}\right)$

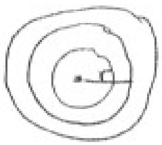
b) $y = \sqrt{(x+1)^{10}(2x+1)^5} = (x+1)^5(2x+1)^{5/2}$

$\Rightarrow \ln y = \ln(x+1)^5(2x+1)^{5/2} = 5\ln(x+1) + \frac{5}{2}\ln(2x+1)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{5}{x+1} + \frac{5}{2}\cdot\frac{1}{2x+1}(2)$

$\Rightarrow \frac{dy}{dx} = \sqrt{(x+1)^{10}(2x+1)^5}\left(\frac{5}{x+1} + \frac{5}{2x+1}\right)$

3)



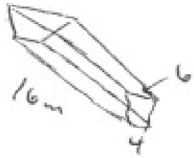
$A = \pi r^2$; $\frac{dr}{dt} = 16 \text{ cm/sec}$

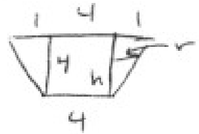
$\Rightarrow \frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$

$\Rightarrow \left.\frac{dA}{dt}\right|_{r=4cm} = 2\pi(4\text{ cm})\cdot 16 \text{ cm/sec}$

$= 128\pi \text{ cm}^2/\text{sec}$

4)





$\frac{h}{4} = \frac{r}{1} \Rightarrow h = 4r$

$\Rightarrow \frac{dh}{dt} = 4\frac{dr}{dt}$

Given: $\frac{dV}{dt} = 10\ m^3/m$

Want: $\frac{dh}{dt}$, when $h = 2m$

$V = 16 \cdot \overset{1}{h} \cdot \frac{4+(4+2r)}{2}$

$= 8 \cdot 4r\,(8+2r)$

$= 256r + 64r^2 \Rightarrow \frac{dV}{dt} = (256 + \overset{\downarrow}{128r})\,\frac{dr}{dt}$

$h=4r \Rightarrow r = \frac{h}{4}$ $\quad \frac{dr}{dt} = \frac{1}{4}\frac{dh}{dt}$

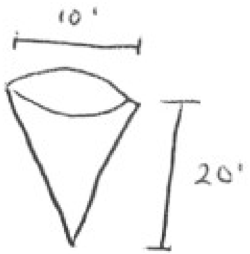
$= (256 + 32h)\cdot\frac{1}{4}\frac{dh}{dt}$

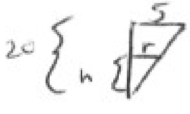
$= (64 + 8h)\,\frac{dh}{dt}$

$\Rightarrow 10\ m^3/min = (64 + 8(2))\cdot\frac{dh}{dt}$

$\Rightarrow \frac{dh}{dt} = \frac{10}{80} = \frac{1}{8}\ m/min$

5)





$h = 16 \Rightarrow r = 4$

$\frac{dV}{dt} = 8\ ft^3/min - \frac{dL}{dt}$ → rate of leaking

$\frac{dh}{dt} = 1\ in/min = \frac{1}{12}\ ft/min$

Want: $\frac{dL}{dt}$

$\frac{h}{20} = \frac{r}{5} \Rightarrow 5h = 20r \Rightarrow h = 4r \Rightarrow \frac{dh}{dt} = 4\frac{dr}{dt} \Rightarrow \frac{dr}{dt}\Big|_{r=4} = \frac{1}{12}$

$= \frac{1}{48}$

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2(4r) = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} \Rightarrow \frac{dV}{dt}\Big|_{r=4} = 4\pi(4)^2\cdot\frac{1}{48}\ ft^3/min$

$= \frac{64}{48}\pi = \frac{4}{3}\pi\ ft^3/min$

when $h = 16ft$, $r = 4f$: $\frac{dL}{dt} = 8 - \frac{dV}{dt} = 8 - \frac{4}{3}\pi\ ft^3/min$

6) spheres $\begin{cases} V = \frac{4}{3}\pi r^3 \\ A = 4\pi r^2 \end{cases}$ $\Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

surface area

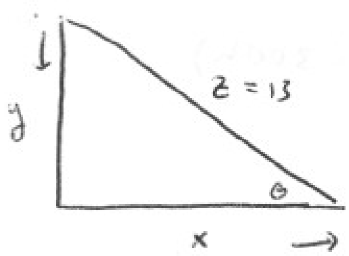
If $\frac{dV}{dt}$ is proportional to A, then by definition of proportional

$$\frac{dV}{dt} = k\, 4\pi r^2 \quad \text{for some constant } k.$$

$$\Rightarrow 4\pi r^2 \frac{dr}{dt} = k \cdot 4\pi r^2$$

$$\Rightarrow \frac{dr}{dt} = k < 0 \quad (\text{since balloon is shrinking})$$

7)



$$x^2 + y^2 = 13^2 = 169$$

$$x = 12, \ \left.\frac{dx}{dt}\right|_{x=12} = 5 \text{ ft/sec}$$

a) $x^2 + y^2 = 169 \Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

when $x = 12$, $y = \sqrt{13^2 - 12^2} = \sqrt{169 - 144} = \sqrt{25} = 5$

If $\frac{dx}{dt} = 5$ ft/sec, then

$$\overset{\text{ft}}{2(12)} \cdot \overset{\text{ft/sec}}{5} + \overset{\text{ft}}{2(5)} \cdot \frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = \frac{-120}{10} = -12 \text{ ft/sec}$$

b) $A = \frac{1}{2}xy \Rightarrow \frac{dA}{dt} = \frac{1}{2}x\frac{dy}{dt} + \frac{1}{2}y\frac{dx}{dt}$

$$\Rightarrow \frac{dA}{dt} = \frac{1}{2}(12)(-12) + \frac{1}{2}(5)(5)$$

$$= \frac{-119}{2} \text{ ft}^2/\text{sec}$$

c) $\tan\theta = \frac{y}{x}$

when $x = 12, y = 5$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{12}{13}} = \frac{13}{12}$$

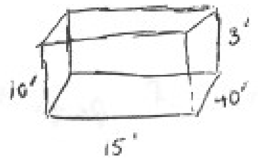
$$\tan\theta = \frac{y}{x} \Rightarrow \sec^2\theta \frac{d\theta}{dt} = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2}$$

when $x = 12, y = 5$

$$\left(\frac{13}{12}\right)\frac{d\theta}{dt} = \frac{12(-12) - (5)(5)}{(12)^2}$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{119}{\cancel{144}_{12}} \cdot \frac{\cancel{12}}{13} = \frac{119}{156}$$

8) (Not to scale)



If $h \le 7$, then

$V = (\frac{1}{2}h)(40)(15)$ $(l \cdot w \cdot h)$

$= 300h$

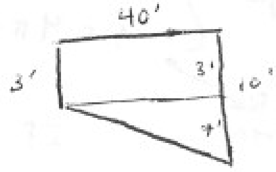
If $h > 7$, then

$V = \frac{7}{2}(40)(15) + (h-7)(15)(40)$

$= 2100 + 600h - 4200$

$= 600h - 2100$

Cross Section:



a) Since $V = 2000 < \frac{7}{2}(40)(15) = 2100$, the pool depth is not above the incline (there is no water in the 3' end.)

Hence, $\frac{dV}{dt} = 300\frac{dh}{dt}$ (Since $V = 300h$)

$\Rightarrow 25\ ft^3/min = 300\frac{dh}{dt}$

$\Rightarrow \frac{dh}{dt} = \frac{1}{12}\ ft/min$

b) Since $V = 3000 > 2100$, the pool depth is above the incline (there is water in the 3' end).

Hence, $\frac{dV}{dt} = 600\frac{dh}{dt}$ (Since $V = 600h - 2100$)

$\Rightarrow 25\ ft^3/min = 600\frac{dh}{dt}$

$\Rightarrow \frac{dh}{dt} = \frac{1}{24}\ ft/min$

9 a) $y = \ln kx \Rightarrow \frac{dy}{dx} = \frac{1}{kx}\overset{\frac{d}{dx}(kx)}{\overset{\shortparallel}{(k)}} = \frac{1}{x}$

b) $y = (\ln x)^3 \Rightarrow \frac{dy}{dx} = 3(\ln x)^2\left(\frac{1}{x}\right) = \frac{3(\ln x)^2}{x}$

c) $y = t\sqrt{\ln t} = t(\ln t)^{1/2} \Rightarrow \frac{dy}{dt} = (1)(\ln t)^{1/2} + t \cdot \frac{1}{2}(\ln t)^{-1/2} \cdot \frac{1}{t}$

$= \sqrt{\ln t} + \frac{1}{2\sqrt{\ln t}}$

d) $y = \ln(\ln(\ln(x))) \Rightarrow \frac{dy}{dx} = \frac{1}{\ln(\ln(x))} \cdot (\ln(\ln x))'$

$= \frac{1}{\ln(\ln(x))} \cdot \frac{1}{\ln x} \cdot (\ln x)'$

$= \frac{1}{\ln(\ln(x))} \cdot \frac{1}{\ln x} \cdot \frac{1}{x}$

e) $y = \sqrt{\frac{(x+1)^5}{(x+2)^{20}}} \Rightarrow \ln y = \ln\left(\frac{(x+1)^5}{(x+2)^{20}}\right)^{1/2}$

$= \frac{1}{2}\left(5\ln(x+1) - 20\ln(x+2)\right)$

$= \frac{5}{2}\ln(x+1) - 10\ln(x+2)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{5}{2(x+1)} - \frac{10}{x+2}$

$\Rightarrow \frac{dy}{dx} = \sqrt{\frac{(x+1)^5}{(x+2)^{20}}}\left(\frac{5}{2(x+1)} - \frac{10}{x+2}\right)$