Math 125, Section 01 Name:________________________________

Exam 2 - March 25, 2008

1. Find $\dfrac{dy}{dx}$ for the following functions.

2.5 for product rule

5 (a) $y = e^{2x}\sin(3x)$

$$\frac{dy}{dx} = \frac{d}{dx}(e^{2x}) \cdot \sin 3x + e^{2x} \cdot \frac{d}{dx}(\sin 3x)$$

$$= 2e^{2x}\sin 3x + 3e^{2x}\cos 3x$$

1/2 1/2 1/2 1/2

5 (b) $y = \dfrac{3x+1}{x^2+1}$

2.5 for QR

2.5 for derivatives

-1.5 if numerator reversed

$$\frac{dy}{dx} = \frac{(x^2+1)(3) - (3x+1)(2x)}{(x^2+1)^2}$$

$$= \frac{3x^2+3-6x^2-2x}{(x^2+1)^2}$$

$$= \frac{-3x^2-2x+3}{(x^2+1)^2}$$

5 (c) $y = \left(1+\dfrac{1}{x}\right)^3$

2.5 2.5

$$\frac{dy}{dx} = 3\left(1+\frac{1}{x}\right)^2 \cdot (-x^{-2})$$

$$= \frac{-3\left(1+\frac{1}{x}\right)^2}{x^2}$$

5 (d) $y = \tan(\cos 2x)$

$$\frac{dy}{dx} = \sec^2(\cos 2x) \cdot \frac{d}{dx}(\cos 2x)$$

$$= \sec^2(\cos 2x) \cdot -\sin 2x \cdot \frac{d}{dx}(2x)$$

$$= -\underset{1}{2}\,\underbrace{\sin 2x}_{2} \cdot \underbrace{\sec^2(\cos 2x)}_{2}$$

2. Find $g'(t)$ for the following functions.

5 (a) $g(t) = \ln(t^3 + 3\sin t)$

$$g'(t) = \frac{3t^2 + 3\cos t \;\; 2}{t^3 + 3\sin t} \qquad 3$$

5 (b) $g(t) = t\ln t - t$

$$4 \qquad\qquad 1$$

$$g'(t) = \boxed{(1)\cdot \ln t + t \cdot \frac{1}{t}} - 1$$

$$= \ln t + 1 - 1$$

$$= \ln t$$

5 (c) $g(t) = \ln\left(\dfrac{t+1}{t^3+1}\right)$

$$2 \qquad\qquad 1$$

$$g'(t) = \left(\frac{t^3+1}{t+1}\right) \cdot \frac{d}{dt}\left(\frac{t+1}{t^3+1}\right)$$

$$= \left(\frac{t^3+1}{t+1}\right) \cdot \frac{(t^3+1)(1) - (t+1)(3t^2)}{(t^3+1)^2} \quad 2$$

$$= \left(\frac{\cancel{t^3+1}}{t+1}\right) \cdot \frac{t^3+1-3t^3-3t^2}{(t^3+1)^{\cancel{2}}}$$

$$= \frac{-2t^3-3t^2+1}{(t+1)(t^3+1)}$$

$g(t) = \ln(t+1) - \ln(t^3+1)$

$$g'(t) = \frac{1}{t+1} - \frac{3t^2}{t^3+1}$$

5 3. Find $f'(x)$ if $f(x) = \sec(e^{2x^3})$. 2

$$f'(x) = \boxed{\sec(e^{2x^3}) \cdot \tan(e^{2x^3})} \cdot \frac{d}{dx}(e^{2x^3})$$

$$= \underbrace{\boxed{6x^2}}_{1} \underbrace{\boxed{e^{2x^3}}}_{2} \sec(e^{2x^3}) \tan(e^{2x^3})$$

6 4. Calculate $f''(x)$ if $f(x) = \sin^2 x$. $= (\sin x)^2$

4 - 2 for product rule, 2 for derivatives

$$f'(x) = 2\sin x \cdot \cos x \quad 2$$

$$\Rightarrow f''(x) = 2 \cdot \frac{d}{dx}(\sin x)\cos x + 2\sin x \cdot \frac{d}{dx}(\cos x)$$

$$= 2 \cdot \cos x \cdot \cos x + 2 \cdot \sin x \cdot (-\sin x)$$

$$= 2\cos^2 x - 2\sin^2 x$$

$$= 2(\cos^2 x - \sin^2 x)$$

6 5. Prove that if $f(x) = \tan x$, then $f'(x) = \sec^2 x$.
(You may use known derivatives of $\sin x$ and $\cos x$.)

$$f(x) = \frac{\sin x}{\cos x} \quad 1 \qquad \Rightarrow f'(x) = \frac{\cos x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot \frac{d}{dx}(\cos x)}{\cos^2 x} \quad 2$$

$$= \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x} \quad 1$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \quad \tfrac{1}{2}$$

$$= \frac{1}{\cos^2 x} \quad 1$$

$$= \sec^2 x \quad \tfrac{1}{2}$$

6. Find the derivatives of each of the following. (Recall $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ and $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$.)

(4) (a) $y = \sin^{-1}\sqrt{x}$

$$y' = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{d}{dx}(x^{1/2}) \quad (2)$$

$$= \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2}x^{-1/2} \quad (2)$$

$$= \frac{1}{2\sqrt{x}\cdot\sqrt{1-x}}$$

(4) (b) $y = x^2 \tan^{-1}\sqrt{x-1}$ (.5)

(1.5) Product rule (2)

$$y' = 2x\tan^{-1}\sqrt{x-1} + x^2 \cdot \frac{1}{1+(\sqrt{x-1})^2}\frac{d}{dx}(x-1)^{1/2}$$

$$= 2x\tan^{-1}\sqrt{x-1} + \frac{x^{\cancel{2}}}{\cancel{1}+x-\cancel{1}} \cdot \frac{1}{2}(x-1)^{-1/2}$$

$$= 2x\tan^{-1}\sqrt{x-1} + \frac{x}{2\sqrt{x-1}}$$

(10) 7. At the end of a basketball game, Lebron James throws the basketball directly upward in the air, and the ball's height is $h(t) = -8t^2 + 24t + 128$ feet after t seconds.

Bad problem (d)

(4) (a) What are ~~been~~ the ball's velocity, speed, and acceleration after t seconds?

(2) $v(t) = h'(t) = -16t + 24$

(1) $s(t) = |v(t)| = |-16t + 24|$

(1) $a(t) = h''(t) = v'(t) = -16 \text{ ft/sec}^2$

(2.5) (b) How long does it take for the ball to reach its maximum height?

$v(t) = 0 \Rightarrow -16t + 24 = 0$

$\Rightarrow 16t = 24$

$\Rightarrow t = \frac{24}{16} = \frac{3}{2}$ sec

(2.5) (c) What is its maximum height?

$$h(\tfrac{3}{2}) = -8(\tfrac{3}{2})^2 + \overset{12}{\cancel{24}}(\tfrac{3}{\cancel{2}}) + 128$$

$$= -\overset{2}{\cancel{8}} \cdot \frac{9}{\cancel{4}} + 36 + 128$$

$$= -18 + 36 + 128$$

$$= 146 \text{ ft}$$

(0) (d) How long does it take before the ball hits the ground?

$t = 2 \cdot (\frac{3}{2}) = 3$ sec

or

$0 = -8t^2 + 24t + 128 = -8(t^2 - 3t - 16) = 0$

$t = \frac{3 \pm \sqrt{9+64}}{2} = \frac{3 \pm \sqrt{73}}{2} \approx 5.77$ sec

(4) 8. Consider the function $f(x) = \sqrt{1+x}$. $= (1+x)^{1/2}$

(5) (a) Find the linearization of f at $x = 3$.

(1) $f'(x) = \frac{1}{2}(1+x)^{-1/2}$

(1) $f'(3) = \frac{1}{2}(4)^{-1/2} = \frac{1}{2(2)} = \frac{1}{4}$

(1) $f(3) = \sqrt{4} = 2$

$y - 2 = \frac{1}{4}(x-3)$

$y = \frac{1}{4}x - \frac{3}{4} + 2$

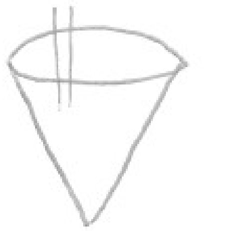
(2) $L(x) = \frac{1}{4}x + \frac{5}{4}$

(2) (b) Use the linearization from part (a) to approximate $\sqrt{4.02}$.

$\sqrt{4.02} \approx L(3.02) = \frac{1}{4}(3.02) + \frac{5}{4}$

$= 2.005$

(7) 9. A (21-year old) student is using a straw to drink beer out of a conical paper cup at a rate of **2** cubic centimeters per second. If the height of the cup is ~~10~~ 12 centimeters and the diameter at the top of the cup is **6** centimeters, how fast is the beer level falling when the depth of the beer is 5 centimeters? (Recall: $V = \frac{1}{3}\pi r^2 h$ for a cone.)

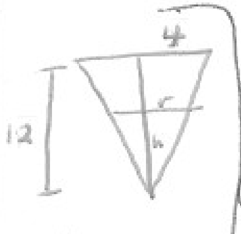


V = volume
r = radius
h = height

Want: $\left.\frac{dh}{dt}\right|_{h=5}$

Know $h = 5 \Rightarrow r = \frac{5}{3}$ (1)

$\frac{dV}{dt} = -2 \text{ cm}^3/\text{sec}$



(2)

$\frac{h}{12} = \frac{r}{4}$

$12r = 4h$

$3r = h$

$3\frac{dr}{dt} = \frac{dh}{dt}$

$V = \frac{1}{3}\pi r^2 h$

$\frac{dV}{dt} = \frac{2}{3}\pi r h \frac{dr}{dt} + \frac{1}{3}\pi r^2 \frac{dh}{dt}$

$= \frac{2}{3}\pi r h \frac{dr}{dt} + \pi r^2 \frac{dr}{dt}$

$-2 = \frac{2}{3}\pi\left(\frac{5}{3}\right)(5)\cdot\frac{dr}{dt} + \pi\left(\frac{25}{9}\right)\frac{dr}{dt}$

$\Rightarrow \frac{dr}{dt} = \frac{-2}{\frac{50\pi}{9} + \frac{25\pi}{9}} = \frac{-2}{\frac{75\pi}{9}}$

$= \frac{-18}{75\pi}$ cm/sec $\Rightarrow \frac{dh}{dt} = \frac{-18}{25\pi}$ cm/sec

$(-.2292)$

(2)

or $V = \frac{1}{3}\pi r^2 \cdot (3r)$

$= \pi r^3$

$\Rightarrow \frac{dV}{dt} = 3\pi r^2 \frac{dr}{dt}$

(3) for plugging in & solving

$-2 = 3\pi\left(\frac{5}{3}\right)^2 \frac{dr}{dt} = \frac{25\pi}{3}\frac{dr}{dt}$

$\Rightarrow \frac{dr}{dt} = \frac{-6}{25\pi}$

(1) for units

$\Rightarrow \frac{dh}{dt} = 3\left(\frac{-6}{25\pi}\right)$

$= \frac{-18}{25\pi}$ cm/sec.

(12) 10. Suppose that the functions f and g and their derivatives with respect to x have the following values at $x = 2$ and $x = -3$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	-3	-5	-1/2	3
-3	1	7	4	-8

Find the derivatives below with respect to x at the given values. (Simplified answers are expected.)

(3) (a) $h'(-3)$, where $h(x) = f(x) \cdot g(x)$

$h'(x) = f(x)\, g'(x) + f'(x)\, g(x)$ (2)

$h'(-3) = f(-3)\, g'(-3) + f'(-3) \cdot g(-3)$ (1)

$= (1)(-8) + (4)(7)$

$= -8 + 28$

$= 20$

(3) (b) $h'(2)$, where $h(x) = 7f(x) - 3g(x)$

$h'(x) = 7f'(x) - 3g'(x)$ (2)

$h'(2) = 7\left(-\frac{1}{2}\right) - 3(3)$ (1)

$= -\frac{7}{2} - 9$

$= \frac{-25}{2}$

(3) (c) $h'(2)$, where $h(x) = g(f(x))$

$h'(x) = g'(f(x)) \cdot f'(x)$ (2)

$\Rightarrow h'(2) = g'(-3) \cdot -\frac{1}{2}$ (1)

$= -8 \cdot -\frac{1}{2}$

$= 4$

(3) (d) $h'(-3)$, where $h(x) = \ln(2x + g(x))$

$h'(x) = \dfrac{2 + g'(x)}{2x + g(x)}$ (2.5)

$\Rightarrow h'(-3) = \dfrac{2 + (-8)}{2(-3) + 7}$

$= \dfrac{-6}{1}$ (.5)

$= -6$

(6) 11. Find the equation of the tangent line to the curve $x^3 + y^3 - 9xy = 0$ at the point $(2, 4)$.

$3x^2 + 3y^2 \frac{dy}{dx} - 9y - 9x\frac{dy}{dx} = 0$ (1.5)

(1) $\frac{dy}{dx}(3y^2 - 9x) = 9y - 3x^2 \Rightarrow \frac{dy}{dx} = \dfrac{9y - 3x^2}{3y^2 - 9x}$ (1) $= \dfrac{3y - x^2}{y^2 - 3x}$

$\Rightarrow \frac{dy}{dx}\Big|_{(2,4)} = \dfrac{36 - 12}{48 - 18} = \dfrac{24}{30} = \dfrac{4}{5}$ (1)

$y - 4 = \frac{4}{5}(x - 2)$

$y - 4 = \frac{4}{5}x - \frac{8}{5}$ (1.5)

$y = \frac{4}{5}x + \frac{12}{5}$