Lab 4

1a) $\int \frac{4x-1}{(x-1)(x+2)}dx$

$= \int \frac{1}{x-1} + \frac{3}{x+2}dx$

$= \ln|x-1| + 3\ln|x+2| + C$

$\frac{A}{x-1} + \frac{B}{x+2} = \frac{4x-1}{(x-1)(x+2)}$

Heaviside Cover-Up: $A = \frac{4(1)-1}{1+2}$

$= 1$

$B = \frac{4(-2)-1}{-2-1} = 3$

b) $\int \frac{1}{(x+a)(x+b)}dx$

$= \int \frac{\frac{1}{-a+b}}{x+a} + \frac{\frac{1}{-b+a}}{x+b}dx$

$= \frac{1}{-a+b}\ln|x+a| + \frac{1}{-b+a}\ln|x+b| + C$

$\frac{A}{x+a} + \frac{B}{x+b} = \frac{1}{(x+a)(x+b)}$

Heaviside Cover-Up) $A = \frac{1}{-a+b}$

$B = \frac{1}{-b+a}$

c) $\int \frac{x^2+1}{x^2-x}dx = \int \frac{x^2+1}{x(x-1)}dx$

$= \int 1 + \frac{x+1}{x(x-1)}dx$

$= \int 1 + \frac{-1}{x} + \frac{2}{x-1}dx$

$= x - \ln|x| + 2\ln|x-1| + C$

$$\begin{array}{r|l} & 1 \\ \hline x^2-x & x^2 \quad +1 \\ & -x^2+x \\ \hline & x+1 \end{array}$$

$\frac{A}{x} + \frac{B}{x-1} = \frac{x+1}{x(x-1)}$

Heavicide Cover-Up: $A = \frac{1}{-1} = -1$

$B = \frac{1+1}{1} = 2$

d) $\int t^2 \ln t\, dt$

$= \frac{1}{3}t^3 \ln t - \int \frac{1}{3}t^3 \cdot \frac{1}{t}dt$

$= \frac{1}{3}t^3 \ln t - \frac{1}{3}\int t^2 dt$

$= \frac{1}{3}t^3 \ln t - \frac{1}{9}t^3 + C$

$u = \ln t$

$du = \frac{1}{t}dt$

$v = \frac{1}{3}t^3$

$dv = t^2 dt$

e) $\int_0^2 x^3 \sqrt{4-x^2}\, dx$

$= \int_{\pi/2}^{0} (2\cos\theta)^3 \cdot 2\sin\theta \cdot (-2\sin\theta)\, d\theta$

$= 32 \int_0^{\pi/2} \cos^3\theta \sin^2\theta\, d\theta$

$= 32 \int_0^{\pi/2} (1-\sin^2\theta)\sin^2\theta \cdot \cos\theta\, d\theta$

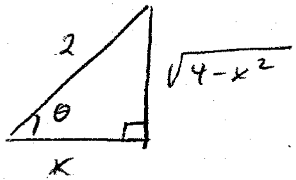
$= 32 \int_0^{\pi/2} (\sin^2\theta - \sin^4\theta)\cos\theta\, d\theta$

$= 32 \int_0^1 u^2 - u^4\, du$

$= 32 \left(\frac{1}{3}u^3 - \frac{1}{5}u^5 \right|_0^1$

$= 32\left(\frac{1}{3} - \frac{1}{5}\right) = 32\left(\frac{5}{15} - \frac{3}{15}\right)$

$= \frac{64}{15}$



$\sin\theta = \frac{\sqrt{4-x^2}}{2} \Rightarrow 2\sin\theta = \sqrt{4-x^2}$

$\cos\theta = \frac{x}{2} \Rightarrow x = 2\cos\theta$

$dx = -2\sin\theta\, d\theta$

$x = 0 \Rightarrow \theta = \cos^{-1}\frac{0}{2} = \frac{\pi}{2}$

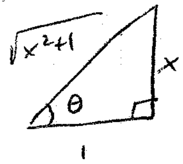
$x = 2 \Rightarrow \theta = \cos^{-1}\frac{2}{2} = 0$

Let $u = \sin\theta \Rightarrow du = \cos\theta\, d\theta$

$\theta = 0 \Rightarrow u = \sin 0 = 0$

$\theta = \frac{\pi}{2} \Rightarrow u = \sin\frac{\pi}{2} = 1$

f) $\int \sqrt{x^2+1}\, dx$



$\tan\theta = \frac{x}{1} \Rightarrow \sec^2\theta\, d\theta = dx$

$\cos\theta = \frac{1}{\sqrt{x^2+1}} \Rightarrow \sec\theta = \sqrt{x^2+1}$

$= \int \sec\theta \cdot \sec^2\theta\, d\theta$

$= \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\, d\theta$

$u = \sec\theta$

$du = \sec\theta\tan\theta\, d\theta$

$v = \tan\theta$

$dv = \sec^2\theta\, d\theta$

$= \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)\, d\theta$

$\int \sec^3\theta\, d\theta = \sec\theta\tan\theta - \int \sec^3\theta\, d\theta + \int \sec\theta\, d\theta = \sec\theta\tan\theta - \int \sec^3\theta\, d\theta + \ln|\sec\theta + \tan\theta|$

$\Rightarrow 2\int \sec^3\theta\, d\theta = \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| + C$

So, $\int \sqrt{x^2+1}\, dx = \frac{1}{2}\left[x\sqrt{x^2+1} + \ln\left|\sqrt{x^2+1} + x\right|\right] + C$

g) $\int_0^1 \frac{\sqrt{t}}{t+1}\,dt$

$u = \sqrt{t} \Rightarrow u^2 = t$

$\Rightarrow 2u\,du = dt$

$t=0 \Rightarrow u=\sqrt{0}=0$

$t=1 \Rightarrow u=\sqrt{1}=1$

$$= \int_0^1 \frac{u}{u^2+1}\cdot 2u\,du$$

$$= 2\int_0^1 \frac{u^2}{u^2+1}\,du$$

$$\begin{array}{r} 1 \\ u^2+1\,\overline{)\,u^2} \\ \underline{-\;u^2+1} \\ -1 \end{array}$$

$\frac{u^2}{u^2+1} = 1 - \frac{1}{u^2+1}$

$$= 2\int_0^1 1 - \frac{1}{u^2+1}\,du$$

$$= 2(u - \tan^{-1} u\Big|_0^1$$

$$= 2\left[\left(1-\tfrac{\pi}{4}\right) - (0-0)\right]$$

$$= 2\left(1-\tfrac{\pi}{4}\right)$$

h) $\int \frac{x}{\sqrt{1-x}}\,dx$

$u = 1-x \Rightarrow x = 1-u$

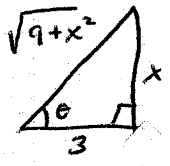
$du = -dx \Rightarrow dx = -du$

$$= -\int \frac{1-u}{\sqrt{u}}\,du$$

$$= \int u^{1/2} - u^{-1/2}\,du$$

$$= \tfrac{2}{3}u^{3/2} - 2u^{1/2} + C \quad = \tfrac{2}{3}(1-x)^{3/2} - 2\sqrt{1-x} + C$$

i) $\int_0^3 \frac{dx}{9+x^2}$



$\tan\theta = \frac{x}{3} \Rightarrow x = 3\tan\theta$

$\Rightarrow dx = 3\sec^2\theta\,d\theta$

$\cos\theta = \frac{3}{\sqrt{9+x^2}}$

$\sec\theta = \frac{\sqrt{9+x^2}}{3}$

$\Rightarrow 9\sec^2\theta = 9+x^2$

$x=0 \Rightarrow \tan\theta = 0 \Rightarrow \theta = 0$

$x=3 \Rightarrow \tan\theta = \frac{3}{3} = 1 \Rightarrow \theta = \frac{\pi}{4}$

$$= \int_0^{\pi/4} \frac{3\cancel{\sec^2\theta}\,d\theta}{9\cancel{\sec^2\theta}\,d\theta}$$

$$= \int_0^{\pi/4} \tfrac{1}{3}\,d\theta$$

$$= \tfrac{1}{3}\theta\Big|_0^{\pi/4}$$

$$= \frac{\pi}{12}$$

$u = \cos\frac{x}{2}$

$du = -\frac{1}{2}\sin\frac{x}{2}\,dx \Rightarrow -2du = \sin\frac{x}{2}dx$

$x = 0 \Rightarrow u = \cos 0 = 1$

$x = \pi \Rightarrow u = \cos\frac{\pi}{2} = 0$

j) $\int_0^\pi \sin^5\frac{x}{2}dx = \int_0^\pi \sin^4\frac{x}{2}\cdot\sin\frac{x}{2}\,dx$

$= \int_0^\pi \left(1-\cos^2\frac{x}{2}\right)^2\cdot\sin\frac{x}{2}dx$

$= \int_1^0 (1-u^2)^2\cdot -2du$

$= 2\int_0^1 1-2u^2+u^4 du$

$= 2(u - \frac{2}{3}u^3 + \frac{1}{5}u^5 \Big|_0^1$

$= 2(1 - \frac{2}{3} + \frac{1}{5}) = 2(\frac{15}{15} - \frac{10}{15} + \frac{3}{15}) = 2(\frac{8}{15}) = \frac{16}{15}$

k) $\int_0^{\pi/2} 7\cos^7 t\,dt$

$= 7\int_0^{\pi/2} \cos^6 t\cdot\cos t\,dt$

$u = \sin t$

$du = \cos t\,dt$

$t = 0 \Rightarrow u = \sin 0 = 0$

$t = \frac{\pi}{2} \Rightarrow u = \sin\frac{\pi}{2} = 1$

$= 7\int_0^{\pi/2} (1-\sin^2 t)^3\cos t\,dt$

$= 7\int_0^1 (1-u^2)^3 du$

$= 7\int_0^1 1-3u^2+3u^4-u^6 du$

$= 7(u - u^3 + \frac{3}{5}u^5 - \frac{1}{7}u^7 \Big|_0^1 = 7(\cancel{1} - \cancel{1} + \frac{3}{5} - \frac{1}{7})$

$= \cancel{7}\left(\frac{21-5}{\cancel{35}\,5}\right) = \frac{16}{5}$

l) $\int_0^\pi 8\sin^4 y\cos^2 y\,dy$

$= 8\int_0^\pi \left(\frac{1-\cos 2y}{2}\right)^2\left(\frac{1+\cos 2y}{2}\right)dy$

$= \int_0^\pi (1-2\cos 2y+\cos^2 2y)(1+\cos 2y)\,dy$

$= \int_0^\pi 1 + \cos 2y - 2\cos 2y - 2\cos^2 2y + \cos^2 2y + \cos^3 2y\,dy$

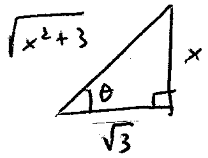
$= \int_0^\pi 1 - \cos 2y - \cos^2 2y + \cos^3 2y\,dy$

See example 3, p. 566-567

$= y - \frac{1}{2}\sin 2y - \frac{1}{2}(y + \frac{1}{4}\sin 4y) + \frac{1}{2}(\sin 2y - \frac{1}{3}\sin^3 2y)\Big|_0^\pi$

$= \pi - 0 - \frac{1}{2}(\pi + 0) + \frac{1}{2}(0-0) = \frac{\pi}{2}$

m) $\int \frac{dx}{x\sqrt{x^2+3}}$



$\tan\theta = \frac{x}{\sqrt{3}} \Rightarrow x = \sqrt{3}\tan\theta$
$\Rightarrow dx = \sqrt{3}\sec^2\theta d\theta$

$\cot\theta = \frac{\sqrt{3}}{x} \Rightarrow \frac{1}{\sqrt{3}}\cot\theta = \frac{1}{x}$

$\cos\theta = \frac{\sqrt{3}}{\sqrt{x^2+3}} \Rightarrow \sqrt{3}\sec\theta = \frac{1}{\sqrt{x^2+3}}$

$= \int \frac{1}{\cancel{\sqrt{3}}}\cot\theta \cdot \cancel{\sqrt{3}}\sec\theta \cdot \sqrt{3}\sec^2\theta d\theta$

$= \sqrt{3}\int \frac{\cancel{\cos\theta}}{\sin\theta} \cdot \frac{1}{\cancel{\cos\theta}} \cdot \frac{1}{\cos^2\theta} d\theta$

$= \sqrt{3}\int \csc\theta \cdot \sec^2\theta d\theta$

$= \sqrt{3}\int \csc\theta \cdot (1+\tan^2\theta) d\theta$

$= \sqrt{3}\int \csc\theta + \frac{1}{\cancel{\sin\theta}} \cdot \frac{\sin^{\cancel{2}}\theta}{\cos^2\theta} d\theta$

$= \sqrt{3}\int \csc\theta d\theta + \sqrt{3}\int \frac{\sin\theta}{\cos\theta} \cdot \frac{1}{\cos\theta} d\theta$

$= -\sqrt{3} \ln|\csc\theta + \cot\theta| + \sqrt{3}\int \sec\theta\tan\theta d\theta$

$= -\sqrt{3} \ln|\csc\theta + \cot\theta| + \sqrt{3} \cdot \sec\theta + C$

$= -\sqrt{3} \ln\left|\frac{\sqrt{x^2+3}}{x} + \frac{\sqrt{3}}{x}\right| + \cancel{\sqrt{3}} \cdot \frac{\sqrt{x^2+3}}{\cancel{\sqrt{3}}} + C$

$= \sqrt{x^2+3} - \sqrt{3} \ln\left|\frac{\sqrt{x^2+3}}{x} + \frac{\sqrt{3}}{x}\right| + C$