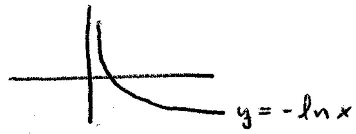
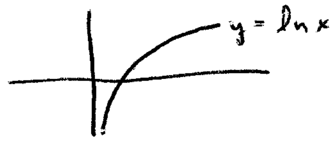
$u = \ln x \qquad v = x$

$du = \frac{1}{x}dx \qquad dv = dx$

1a) $\displaystyle\int_0^1 (-\ln x)\,dx = -\lim_{a\to 0^+} \int_a^1 \ln x\,dx$



$\displaystyle = -\lim_{a\to 0^+} \left(x\ln x \Big|_a^1 - \int_a^1 \cancel{x}\cdot \frac{1}{\cancel{x}}\,dx \right)$



$\displaystyle = -\lim_{a\to 0^+} \quad (1)\ln 1 - a\ln a - x\Big|_a^1$

$\displaystyle = -\lim_{a\to 0^+} \quad 0 - a\ln a - 1 + a^{\nearrow 0}$

So, we must determine $\displaystyle\lim_{a\to 0^+} a\ln a = \lim_{a\to 0^+} \frac{\ln a}{\frac{1}{a}} \qquad \frac{-\infty}{\infty}$

$\displaystyle = \lim_{a\to 0^+} \frac{\frac{1}{a}}{\frac{-1}{a^2}}$ (L'Hopital's Rule)

$\displaystyle = \lim_{a\to 0^+} \frac{-a^{\cancel{2}}}{\cancel{a}} = 0$

So, $\displaystyle\int_0^1 (-\ln x)\,dx = -\lim_{a\to 0^+} \; -a\ln a - 1 + a = -1$ (The integral converges)

b) $\displaystyle\int_0^1 \frac{4r}{\sqrt{1-r^4}}$ (Vertical asymptote at $r=1$, $u=1$) Let $u = r^2$ $\Rightarrow du = 2r\,dr$ $\Rightarrow 2du = 4r\,dr$ $r=0 \Rightarrow u = 0^2 = 0$ $r=1 \Rightarrow u = 1^2 = 1$

$\displaystyle = \int_0^1 \frac{2\,du}{\sqrt{1-u^2}}$

$\displaystyle = \lim_{a\to 1^-} \int_0^a \frac{2\,du}{\sqrt{1-u^2}}$

$\displaystyle = \lim_{a\to 1^-} \quad 2\sin^{-1} u \Big|_0^a$

$\displaystyle = \lim_{a\to 1^-} \quad 2\sin^{-1} a - 2\sin^{-1} 0$

$\displaystyle = 2\left(\frac{\pi}{2}\right) - 2(0)$

$= \pi$ (The integral converges.)

$u = \ln x$
$du = \frac{1}{x}dx$
$v = x$
$dv = dx$

2a) $\int_{-1}^{1} \ln|x|\,dx = \int_{-1}^{0} \ln(-x)\,dx + \int_{0}^{1} \ln x\,dx$

$u = \ln(-x)$ $\quad v = x$
$du = \frac{1}{-x}(-1)dx$ $\quad dv = dx$

$$= \lim_{a\to 0^-} \int_{-1}^{a} \ln(-x)dx + \lim_{b\to 0^+} \int_{b}^{1} \ln x\,dx$$

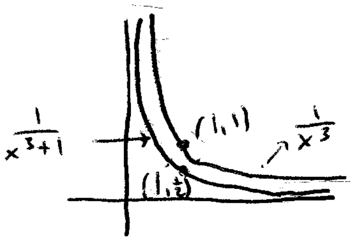
$$= \lim_{a\to 0^-} x\ln(-x)\Big|_{-1}^{a} - \int_{-1}^{a} x\cdot\frac{1}{x}dx + \lim_{b\to 0^+} x\ln x\Big|_{b}^{1} - \int_{b}^{1} x\cdot\frac{1}{x}dx$$

$$= \lim_{a\to 0^-} a\ln(-a) + \underbrace{\ln 1}_{=0} - x\Big|_{-1}^{a} + \lim_{b\to 0} \underbrace{\ln 1}_{=0} - b\ln b - x\Big|_{b}^{1}$$

$$= \lim_{a\to 0^-} a\ln(-a) - a + 1 + \lim_{b\to 0} -b\ln b - 1 + b$$

(by 1a) $\quad = 0 - 0 + 1 + 0 - 1 + 0 = 0$ (Integral converges)

b) $\int_{1}^{\infty} \frac{dx}{x^3+1}$



We know that $\int_{1}^{\infty} \frac{1}{x^3}dx$ converges $(p = 3 > 1)$

Since $\frac{1}{x^3+1} < \frac{1}{x^3}$ on $[1,\infty)$, we know

$\int_{1}^{\infty} \frac{dx}{x^3+1}\,dx$ also converges by the Comparison Test.

c) $\int_{\pi}^{\infty} \frac{2+\cos x}{x}\,dx$

We know that $\int_{\pi}^{\infty} \frac{1}{x}dx$ diverges $(p=1)$.

$2 + \cos x \geq 2 + (-1) = 1$, since $\cos x \geq -1$ for all x

we know $\frac{2+\cos x}{x} \geq \frac{1}{x}$.

Hence, $\int_{\pi}^{\infty} \frac{2+\cos x}{x}dx$ also diverges by the Comparison Test.

d) $\int_{1}^{\infty} \frac{\sqrt{x+1}}{x^2}\,dx$

We know that $\int_{1}^{\infty} \frac{1}{x^{3/2}}dx$ converges $(p = \frac{3}{2} > 1)$

$$\lim_{x\to\infty} \frac{\frac{\sqrt{x+1}}{x^2}}{\frac{1}{x^{3/2}}} = \lim_{x\to\infty} \frac{\sqrt{x+1}}{x^2}\cdot\sqrt{x^3} = \lim_{x\to\infty} \frac{\sqrt{x^4+x^3}}{x^2}\cdot\frac{\frac{1}{\sqrt{x^4}}}{\frac{1}{x^2}} \quad \left(\sqrt{x^4} = x^2\right)$$

$$= \lim_{x\to\infty} \frac{\sqrt{1+\frac{1}{x}}}{1} = 1 < \infty$$

So, $\int_{1}^{\infty} \frac{\sqrt{x+1}}{x^2}dx$ also converges by the Limit Comparison Test.

e) $\int_1^\infty \frac{e^x}{x}\,dx$

We know that $\int_1^\infty \frac{1}{x}\,dx$ diverges, and

$x \geq 1 \Rightarrow e^x \geq e^1 > 1$, so $\frac{e^x}{x} \geq \frac{1}{x}$

Hence, $\int_1^\infty \frac{e^x}{x}\,dx$ also diverges by the Comparison Test.

f) $\int_1^\infty \frac{x}{e^x}\,dx = \lim_{a\to\infty} \int_1^a xe^{-x}\,dx$

$u = x$ $\quad v = -e^{-x}$

$du = dx$ $\quad dv = e^{-x}dx$

$= \lim_{a\to\infty} -xe^{-x}\Big|_1^a + \int_1^a e^{-x}\,dx$

$= \lim_{a\to\infty} -ae^{-a} + e^{-1} - e^{-x}\Big|_1^a$

$= \lim_{a\to\infty} \frac{-a}{e^a} + e^{-1} - e^{-a} + e^{-1}$

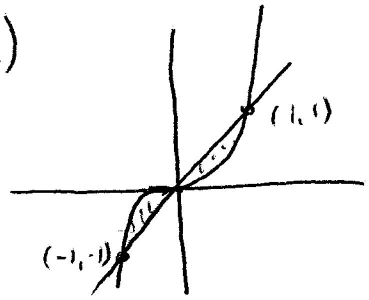
$\lim_{a\to\infty} \frac{-a}{e^a} = \lim_{a\to\infty} \frac{-1}{e^a}$ (L'Hopital)

$= 0$

$= 0 + e^{-1} - 0 + e^{-1}$

$= \frac{2}{e}$ (Integral converges)

3a)



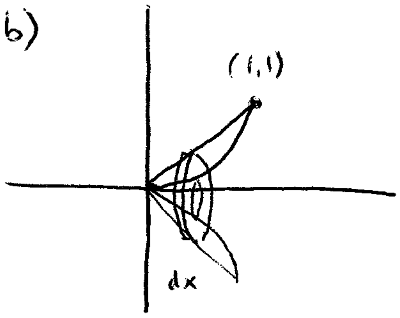
$\text{Area} = \int_{-1}^0 x^3 - x\,dx + \int_0^1 x - x^3\,dx$

$= 2\int_0^1 x - x^3\,dx$

$= 2\left(\frac{1}{2}x^2 - \frac{1}{4}x^4\right|_0^1$

$= 2\left(\frac{1}{2} - \frac{1}{4}\right) = 2\left(\frac{1}{4}\right) = \frac{1}{2}$

b)



$\text{Volume} = \int_0^1 \pi\left(x^2 - (x^3)^2\right)dx$

$= \pi\int_0^1 x^2 - x^6\,dx$

$= \pi\left(\frac{1}{3}x^3 - \frac{1}{7}x^7\right|_0^1$

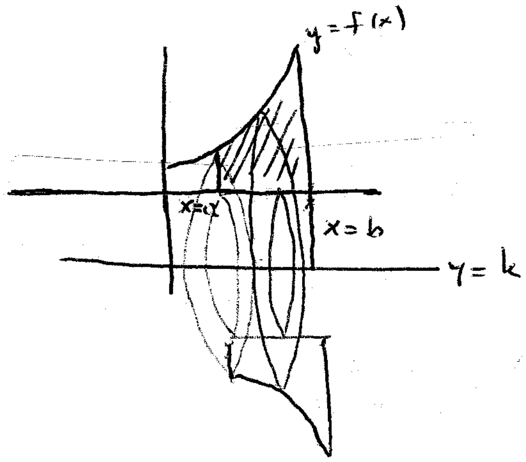
$= \pi\left(\frac{1}{3} - \frac{1}{7}\right)$

$= \frac{4}{21}\pi$

4a) $A(x) = \pi\left[(f(x)-k)^2 - (0-k)^2\right]$

$= \pi\left[(f(x)-k)^2 - k^2\right]$

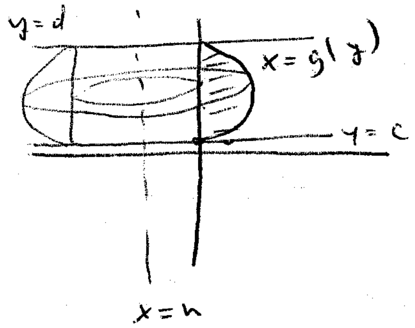
$V = \int_a^b \pi\left[(f(x)-k)^2 - k^2\right]dx$



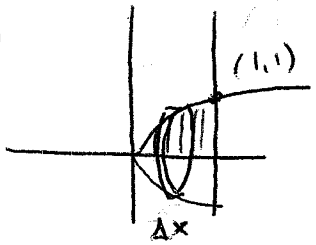
b) $A(y) = \pi\left[(g(y)-h)^2 - (0-h)^2\right]$

$= \pi\left[(g(y)-h)^2 - h^2\right]$

$V = \int_c^d \pi\left[(g(y)-h)^2 - h^2\right]dy$



5a)



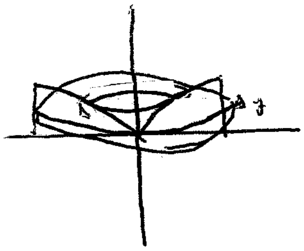
$A(x) = \pi(\sqrt{x})^2$

$V = \int_0^1 \pi(\sqrt{x})^2 dx = \pi\int_0^1 x\,dx$

$= \frac{1}{2}\pi x^2 \Big|_0^1$

$= \frac{1}{2}\pi$

b)

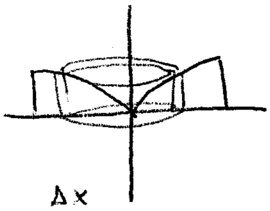


$y = \sqrt{x} \Rightarrow x = y^2$

Washers:

$V = \int_0^1 \pi\left(1^2 - (y^2)^2\right)dy$

$= \pi\int_0^1 1 - y^4\,dy = \pi\left(y - \frac{1}{5}y^5\Big|_0^1\right) = \frac{4}{5}\pi$



Shells

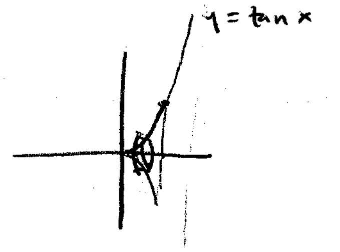
$V = 2\pi\int_0^1 x\cdot\sqrt{x}\,dx$

$= 2\pi\int_0^1 x^{3/2}dx$

$= 2\pi\cdot\frac{2}{5}x^{5/2}\Big|_0^1$

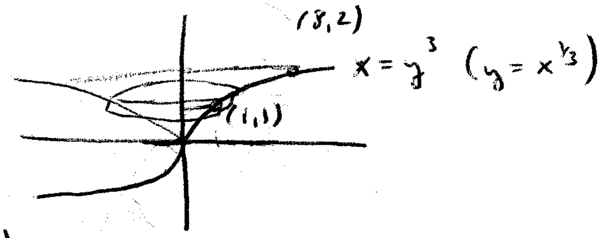
$= \frac{4}{5}\pi$

6 a) $\pi \int_0^{\pi/4} \tan^2 x\,dx$



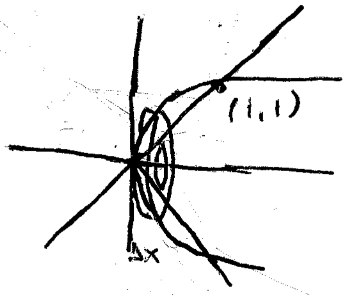
The region between $y = \tan x$ and the x-axis (and the line $x = \frac{\pi}{4}$) is rotated about the x-axis. (Disks are used)

b) $\pi \int_1^2 y^6\,dy$



The region bounded by $x = y^3$, the y-axis, and the lines $y = 1$ and $y = 2$ is rotated about the y-axis. (Disks are used)

c) $\pi \int_0^1 (x - x^2)\,dx = \pi \int_0^1 (\sqrt{x})^2 - (x)^2\,dx$



The region in the first quadrant that is bounded by $y = \sqrt{x}$ and $y = x$ is rotated about the x-axis. (Washers are used)