Lab 10

1a) $\sum_{n=0}^{\infty} (-1)^n \frac{7}{6^n} = 7 - \frac{7}{6} + \frac{7}{36} - \frac{7}{216} + \cdots$

$\to a = 7$

$\sum_{n=0}^{\infty} 7 \cdot \left(\frac{-1}{6}\right)^n \to r = \frac{-1}{6}$ (Geometric series, $a = 7, r = \frac{-1}{6}$)

$|r| < 1 \Rightarrow$ Converges to $\frac{7}{1 - \frac{-1}{6}} = \frac{7}{\frac{7}{6}}$

$= 6$

b) $\sum_{n=1}^{\infty} \frac{9}{n(n+3)}$

$= \sum_{n=1}^{\infty} \frac{3}{n} + \frac{-3}{n+3}$

$= \left(3 - \cancel{\frac{3}{4}}\right) + \left(\frac{3}{2} - \cancel{\frac{3}{5}}\right) + \left(\frac{3}{3} - \cancel{\frac{3}{6}}\right) + \left(\cancel{\frac{3}{4}} - \frac{3}{7}\right)$

$+ \left(\cancel{\frac{3}{5}} - \frac{3}{8}\right) + \left(\cancel{\frac{3}{6}} - \frac{3}{9}\right) + \cdots$

(Telescoping Series)

$= 3 + \frac{3}{2} + \frac{3}{3}$

$= \frac{11}{2}$

$\frac{A}{n} + \frac{B}{n+3} = \frac{9}{n(n+3)}$

$\Rightarrow A(n+3) + Bn = 9$

$n = -3 \Rightarrow -3B = 9 \Rightarrow B = -3$

$n = 0 \Rightarrow 3A = 9 \Rightarrow A = 3$

2a) $\sum_{n=0}^{\infty} \frac{5}{3^n} = 5 + \frac{5}{3} + \frac{5}{9} + \frac{5}{27} + \cdots$

$\to a = 5$

$\sum_{n=0}^{\infty} 5 \cdot \left(\frac{1}{3}\right)^n \to r = \frac{1}{3}$

$r = \frac{1}{3} < 1$

$\Rightarrow$ Converges to $\frac{5}{1 - \frac{1}{3}} = \frac{5}{\frac{2}{3}}$

$= \frac{15}{2}$

b) $\sum_{n=0}^{\infty} \left(\frac{5}{3}\right)^n \Rightarrow r = \frac{5}{3} > 1 \Rightarrow$ Series Diverges)

3 a) $\sum_{n=1}^{\infty} \frac{n}{n+1}$

$\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$, so the series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges by the Test for Divergence

b) $\sum_{n=1}^{\infty} \frac{\ln n}{n+1}$

We know $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ diverges (Harmonic Series),

so $\sum_{n=1}^{\infty} \frac{1}{n+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ also diverges, as it's just the Harmonic Series w/o the first term.

For $n \geq 3$, $\ln n > 1$, so $\frac{\ln n}{n+1} > \frac{1}{n+1}$

Hence, $\sum_{n=1}^{\infty} \frac{\ln n}{n+1}$ also diverges by the Comparison Test.

c) $\sum_{n=1}^{\infty} \frac{5n-3}{n^2-2n+5}$

Looking at the highest degree terms, it appears that the series behaves like $\sum_{n=1}^{\infty} \frac{5n}{n^2} = 5\sum_{n=1}^{\infty} \frac{1}{n}$ for large values of n.

$5\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (Harmonic Series)

$$\lim_{n\to\infty} \frac{\frac{5n-3}{n^2-2n+5}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{5n^2-3n}{n^2-2n+5} = 5 \qquad (0 < \overset{L}{5} < \infty),$$

so $\sum_{n=1}^{\infty} \frac{5n-3}{n^2-2n+5}$ also diverges by the Limit Comparison Test

d) $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n+\sqrt{n^2+4}}$ Compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$, which is the divergent harmonic series.

$$\lim_{n\to\infty} \frac{\frac{5}{n+\sqrt{n^2+4}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{5n}{n+\sqrt{n^2+4}} \cdot \frac{1/n}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{5}{1+\sqrt{\frac{n^2}{n^2}+\frac{4}{n^2}}}$$

$$= \lim_{n\to\infty} \frac{5}{1+\sqrt{1+\underbrace{\tfrac{4}{n^2}}_{\to 0}}} = \frac{5}{1+\sqrt{1}} = \frac{5}{2} \qquad (0 < \tfrac{5}{2} < \infty)$$

Hence, $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n+\sqrt{n^2+4}}$ also diverges by the Limit Comparison Test.

e) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n+3}$ $\displaystyle\lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2} \neq 0$,

So $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n+3}$ diverges by the Test for Divergence.

f) $\displaystyle\sum_{n=1}^{\infty} \sin\frac{(2n-1)\pi}{2}$

$\left\{\sin\frac{(2n-1)\pi}{2}\right\}_{n=1}^{\infty} = 1, -1, 1, -1, 1, -1, \ldots$, which is a sequence that diverges.

Hence, $\displaystyle\lim_{n\to\infty} \sin\frac{(2n-1)\pi}{2}$ DNE $(\neq 0)$, and consequently

$\displaystyle\sum_{n=1}^{\infty} \sin\frac{(2n-1)\pi}{2}$ diverges by the Test for Divergence.