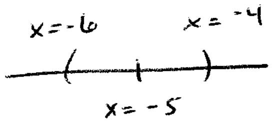
Lab 9

1a) $\sum_{n=0}^{\infty} (x+5)^n$

$\lim_{n\to\infty} \frac{|x+5|^{n+1}}{|x+5|^n} = |x+5| < 1 \Rightarrow R = 1$



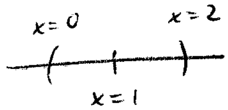
$\Rightarrow$ Converges absolutely on $(-6,-4)$

$x=-6 \Rightarrow \sum_{n=0}^{\infty} (-1)^n$ Diverges by Test for Divergence since $\lim_{n\to\infty} (-1)^n$ DNE.

$x=-4 \Rightarrow \sum_{n=0}^{\infty} (1)^n$ Diverges by Test for Divergence since $\lim_{n\to\infty} 1^n = 1 \neq 0$

Interval of Convergence: $(-6,-4)$

b) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$



$\lim_{n\to\infty} \frac{\frac{|x-1|^{n+1}}{\sqrt{n+1}}}{\frac{|x-1|^n}{\sqrt{n}}} = \lim_{n\to\infty} \frac{|x-1|^{\cancel{n+1}}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{\cancel{|x-1|^n}}$

$= \lim_{n\to\infty} |x-1| \cdot \frac{\sqrt{n}}{\sqrt{n+1}} = |x-1| < 1 \Rightarrow R = 1$

$\Rightarrow$ Converges absolutely on $(0,2)$.

$x=0 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ Converges by AST since $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

$x=2 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ Divergent p-series, $p = \frac{1}{2} < 1$.

Interval of Convergence: $[0,2)$

c) $\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$

$\lim_{n\to\infty} \frac{\left|\frac{3^{n+1} x^{n+1}}{(n+1)!}\right|}{\left|\frac{3^n x^n}{n!}\right|} = \lim_{n\to\infty} \frac{3^{\cancel{n+1}} \cdot |x|^{\cancel{n+1}}}{(n+1)!} \cdot \frac{n!}{\cancel{3^n} \cdot \cancel{|x|^n}}$

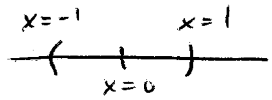
$= \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1 \Rightarrow R = \infty$

$\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$ converges absolutely on $(-\infty, \infty)$.

d) $\sum_{n=1}^{\infty} \frac{(2x+3)^{2n+1}}{n!}$

$\lim_{n\to\infty} \frac{|2x+3|^{2n+3}}{(n+1)!} \cdot \frac{n!}{|2x+3|^{2n+1}} = \lim_{n\to\infty} \frac{|2x+3|^2}{n+1} = 0 < 1 \Rightarrow R = \infty$

$\sum_{n=1}^{\infty} \frac{(2x+3)^{2n+1}}{n!}$ converges absolutely on $(-\infty,\infty)$



e) $\sum_{n=1}^{\infty} (\ln n) x^n$

$$\lim_{n\to\infty} \left| \frac{(\ln(n+1))x^{n+1}}{(\ln n)\cdot x^n} \right| = \lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} \cdot |x| = |x| < 1 \Rightarrow R = 1$$

$x = -1 \Rightarrow \sum_{n=1}^{\infty} (-1)^n \ln n$ diverges since $\lim_{n\to\infty} (-1)^n \ln n$ DNE (Test for Divergence)

$x = 1 \Rightarrow \sum_{n=1}^{\infty} (1)^n \ln n$ diverges since $\lim_{n\to\infty} 1^n \ln n = \infty \neq 0$

(Test for Divergence)

f) $\sum_{n=0}^{\infty} n!\,(x-4)^n$

$$\lim_{n\to\infty} \left| \frac{(n+1)!\,(x-4)^{n+1}}{n!\,(x-4)^n} \right| = \lim_{n\to\infty} (n+1)\,|x-4| = \infty > 1$$

$\Rightarrow R = 0$

Converges only when $x = 4$.

2a) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = \ln 2$

Taylor Series for $\ln x$ centered at $x = 1$

$f(x) = \ln x \Rightarrow f(1) = \ln 1 = 0$

$f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1$

$f''(x) = \frac{-1}{x^2} \Rightarrow f''(1) = -1$

$f'''(x) = \frac{2}{x^3} \Rightarrow f'''(1) = 2$

$f^{(4)}(x) = \frac{-6}{x^4} \Rightarrow f^{(4)}(1) = -6$

$f^{(5)}(x) = \frac{24}{x^5} \Rightarrow f^{(5)}(1) = 24$

$\vdots$

$f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{x^n} \Rightarrow f^{(n)}(1) = (-1)^{n+1}(n-1)!$

$$\Rightarrow \ln x = \sum_{n=1}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{n!}(x-1)^n$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$$

$$\Rightarrow \ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(2-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

3) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots + \frac{x^n}{n!} + \cdots$

a) $f(x) = e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$

$= 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \cdots + \frac{(-1)^n x^n}{n!} + \cdots$

b) $g(x) = e^{3x} = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$

$= 1 + 3x + \frac{9x^2}{2} + \frac{27x^3}{6} + \frac{81x^4}{24} + \cdots + \frac{3^n x^n}{n!} + \cdots$

c) $h(x) = xe^x = x \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x \cdot x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$

$= x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24} + \cdots + \frac{x^{n+1}}{n!} + \cdots$

d) $f(x) = e^{2x} + e^{-2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} + \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!}$

$= \left(1 + \cancel{2x} + \frac{4x^2}{2} + \cancel{\frac{8x^3}{6}} + \frac{16x^4}{24} + \cdots + \frac{2^n x^n}{n!} + \cdots\right)$

$+ \left(1 - \cancel{2x} + \frac{4x^2}{2} - \cancel{\frac{8x^3}{6}} + \frac{16x^4}{24} + \cdots + \frac{(-1)^n 2^n x^n}{n!} + \cdots\right)$

$= 2 + 2\left(\frac{4x^2}{2}\right) + 2\left(\frac{16x^4}{24}\right) + \cdots + 2\left(\frac{2^{2n} x^{2n}}{(2n)!}\right) + \cdots$

only "even" terms remain

$= \sum_{n=0}^{\infty} \frac{2^{2n+1} x^{2n}}{(2n)!}$

4a) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$\Rightarrow e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$

b) $e^{2x} = e^x \cdot e^x$

$= \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \right) \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \right)$

$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots \right) \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots \right)$

$= 1 + x + x + \frac{x^2}{2} + x^2 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^3}{2} + \frac{x^3}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^4}{6} + \frac{x^4}{4} + \frac{x^4}{6} + \frac{x^4}{24} + \cdots$

$= 1 + 2x + \frac{4x^2}{2} + \frac{8x^3}{6} + \frac{16x^4}{24} + \cdots$

$= \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$