Section 11.3 - The Integral Test

1. Given a series $\sum a_n$, we have 2 questions:

   (a) Does it converge?

   (b) If it converges, what does the sum converge to?

   In this section, we answer the second question by comparing $\sum_{n=1}^{\infty} a_n$ to $\int_1^{\infty} f(x)\ dx$, where $f(n) = a_n$ for all natural numbers n.

2. The Integral Test: (Illustrate this.)
   Suppose $c \geq 0$ and $f(x)$ is a decreasing positive function, defined for all $x \geq c$, with $a_n = f(n)$ for all n. (Note: This means $a_n > 0$ for all n.)

   - If $\int_c^{\infty} f(x)\ dx$ converges, then $\sum a_n$ converges.
   - If $\int_c^{\infty} f(x)\ dx$ diverges, then $\sum a_n$ diverges.

   pf. (Converse of Riemann sums-construct a function from rectangles.)
   We'll prove for the case where $c = 1$ but other cases follow accordingly.

   - Let f be a continuous, decreasing positive function for which $a_n = f(n)$.
     Note: For any such f, you can construct a sequence a_n, and for any sequence a_n, you can construct such a function f.
   - $\int_1^{n+1} f(x)\ dx \leq a_1 + a_2 + \cdots + a_n$, since left-endpoints overestimate area.
   - $a_1 + a_2 + \cdots a_n \leq a_1 + \int_1^{n} f(x)\ dx$ (Sketch, right-endpoints underestimate area).
   - Hence, $\int_1^{n+1} f(x)\ dx \leq a_1 + a_2 + \cdots + a_n \leq a_1 + \int_1^{n} f(x)\ dx$ for all n.
   - Since inequalities hold as $n \to \infty$, if $\int_1^{\infty} f(x)\ dx$ is finite, then right-hand inequality shows $\sum a_n$ is finite.
   - If $\int_1^{\infty} f(x)\ dx$ is infinite, the left-hand inequality shows that $\sum a_n$ is infinite.

3. Example #1: Show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

4. Example #2: Show that the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{3}{3^p} + \cdots$$

   converges if $p > 1$ and diverges if $p \leq 1$.

5. Example #3: Use the integral test to determine whether the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

converges or diverges.

6. Example #4: (Use only if time permits)
Consider the following grouping of terms in the harmonic series:

$$1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}\right) + \cdots$$

(a) Show that the sum of each group of fractions is more than 1/2.

(b) Explain why this shows that the harmonic series does not converge.