Section 11.4 - Comparison Tests

1. Comparison Test: (Analogous to Direct Comparison Test for Definite Integrals)
   Suppose $0 \leq a_n \leq b_n$ for all n.

   - If $\sum b_n$ converges, then $\sum a_n$ converges.
   - If $\sum a_n$ diverges, then $\sum b_n$ diverges.

2. Example #1: Use the comparison test to determine whether the following series converge.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2}}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1+\cos n}{n^2}$

3. Limit Comparison Test:
   Suppose that $a_n > 0$ and $b_n > 0$ for all $n > N$ in a sequence. Then,

   (a) If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

   (b) If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ also converges.

   (c) If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ also diverges.

   pf. of part (a)

   - Since $L/2 > 0$, there exists an integer N such that $n > N$ implies $\left|\frac{a_n}{b_n} - L\right| < \frac{L}{2}$.
   - So, for $n > N$ $\quad -\frac{L}{2} < \frac{a_n}{b_n} - L < \frac{L}{2}$.
   - $\frac{L}{2} < \frac{a_n}{b_n} < \frac{3L}{2}$
   - $\frac{L}{2} \cdot b_n < a_n < \frac{3L}{2} \cdot b_n$
   - Result follows by the (Direct) Comparison Test.

4. Example: Which of the following series converge, and which diverge?

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{3/2}}$ (p. 765, pr. 14)

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(1+\ln n)^2}$ (p. 765, pr. 16)