Section 11.5 - The Ratio and Root Tests

Note: I'll explain but not prove tests-book proves.

1. The Ratio Test:
   For a series $\sum a_n$, suppose
   $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L.$$
   (a) If $L < 1$, then $\sum a_n$ converges.
   (b) If $L > 1$, or if L is infinite, then $\sum a_n$ diverges.
   (c) If $L = 1$, then the test is inconclusive.

   pf. (a)

   - Let r be a number between L and $1 \Rightarrow \epsilon = r - L > 0$.
   - Then, there is an N s.t. for $n > N$, $\frac{a_{n+1}}{a_n} < L + \epsilon = r$.
   - It follows that $a_{N+m} < ra_{N+m-1} < r^2 a_{N+m-2} < \cdots < r^{m-1} a_{N+1} < r^m a_N$.
   - Consider $\sum c_n$, where $c_n = a_n$ for $n = 1, 2, \ldots, N$, and $c_{N+1} = ra_N, c_{N+2} = r^2 a_N, \ldots$.
   - Hence, $\sum_{n=1}^{\infty} c_n = a_1 + a_2 + \cdots + a_{N-1} + a_N(1 + r + r^2 + \cdots)$.
   - It follows that $\sum c_n$ converges because the geometric converges.

   pf. (c)
   The series $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ verify this.

2. Example #1: Determine whether the following series converge or diverge.
   (a) $\sum_{n=1}^{\infty} \frac{3^n + 7}{5^n}$
   (b) $\sum_{n=1}^{\infty} \frac{n!}{10^n}$
   (c) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

3. The Root Test:
   Let $\sum a_n$ be a series with $a_n \geq 0$ for $n \geq N$, and suppose that
   $$\lim_{n\to\infty} \sqrt[n]{a_n} = L.$$
   Then,

   (a) If $L < 1$, the series converges.
   (b) If $L > 1$ or L is infinite, the series diverges.
   (c) If $L = 1$, the test is inconclusive.

pf. (b) If $\lim_{n\to\infty} \sqrt[n]{a_n} = \text{Ł} > 1$, then $\lim_{n\to\infty} a_n \neq 0$. Hence the series diverges by the test for divergence.

pf. (c) The series $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ verify this.

4. Example #2: Determine whether the following series converge or diverge.

   (a) $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$

   (b) $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$

   (c) $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$