Section 11.8 - Taylor and Maclaurin Series

1. In Section 11.7, we say that within the interval of convergence, the sum of a power series is continuous with derivatives of all orders.
   Question: If a function $f(x)$ has derivatives of all orders on I, can it be expressed as a power series?

2. Linear Approximation $f(x) \approx f(0) + f'(0)x$
   Example: $f(x) = \cos x$

3. Taylor Polynomials:

   - If we want to approximate a function $f(x)$ by quadratic $P_2(x)$, we want the derivatives of f and P_2 to agree.
   - Hence, if $P_2(x) = a_0 + a_1x + a_2x^2$, it follows that we want
     (a) $P_2(0) = a_0 = f(0)$
     (b) $P_2'(0) = a_1 = f'(0)$
     (c) $P_2''(0) = 2a_2 = f''(0) \Rightarrow a_2 = \dfrac{f''(0)}{2}$

     In other words, we want $P_2(x) = f(0) + f'(0)x + \dfrac{f''(0)x^2}{2}$. (Quadratic approximation)
   - Example #1: $f(x) = \cos x$
   - If we want to approximate $f(x)$ by an n^{th} degree polynomial $P_n(x)$, we need
     (a) $P_n(0) = a_0 = f(0)$
     (b) $P_n'(0) = a_1 = f'(0)$
     (c) $P_n''(0) = 2a_2 = f''(0) \Rightarrow a_2 = \dfrac{f''(0)}{2}$
     (d) $P_n'''(0) = 6a_3 = f'''(0) \Rightarrow a_3 = \dfrac{f'''(0)}{6}$

     $\vdots$

     (e) $P_n^{(n)}(0) = n!a_n = f^{(n)}(0) \Rightarrow a_n = \dfrac{f^{(n)}(0)}{n!}$

     In other words, we want $P_n(x) = f(0) + f'(0)x + \dfrac{f''(0)}{2!}x^2 + \dfrac{f'''(0)}{3!}x^3 + \cdots + \dfrac{f^{(n)}(0)}{n!}x^n$.
     $f(x) \approx P_n(x)$, and we call $P_n(x)$ the Taylor polynomial of degree n centered at $x = 0$.

4. Example #2:
   Construct the Taylor polynomial of degree n approximating $f(x) = \dfrac{1}{1-x}$ for x near 0.

5. Taylor Polynomial of Degree n Approximating $f(x)$ for x near a

   $$f(x) \approx P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

   The Maclaurin series generated by f is the Taylor series about $x = 0$ and is often just called the Taylor series of f.

6. Example #3:
   Construct the Taylor polynomial of degree 3 approximating the function $f(x) = \ln x$ for x near 1.

7. Example #4: Find the Taylor series for $f(x) = e^x$ centered at $x = 0$, and determine the radius of convergence.

8. Example #5: Determine the Maclaurin series for $f(x) = e^{2x}$.

9. Example #6: Find the Taylor series generated by $f(x) = 2x^3 + x^2 + 3x - 8$ at $x = 1$.

10. Example #7:
    Show how you can use the Taylor approximation $\sin x \approx x - \dfrac{x^3}{3!}$, for x near 0 to explain why $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$.