Section 11.9 - Convergence of Taylor Series; Error Estimates

1. Common Taylor Series expansions (about $x = 0$)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^k}{k!} + \cdots$$

2. Taylor's Formula: (Generalization of the Mean Value Theorem)
   If has derivatives of all orders in an open interval I containing a, then for each $n \in \mathbb{Z}^+$ and for each $x \in I$,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x),$$

   where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

   for some c between a and x. ($R_n(x)$ is called the remainder or error term.)

3. Note that for $f(x) = e^x$, $R_n(x) = \dfrac{e^c}{(n+1)!}x^{n+1} < e^x \dfrac{x^{n+1}}{(n+1)!} \to 0$
   Inequality is true since $0 < c < x$ and e^x is increasing for all $x > 0$.

4. The Remainder Estimation Theorem:
   If $f^{n+1}(t) \le M$ for all $t \in [x, a]$, then

$$R_n(x) \le M \frac{|x-a|^{n+1}}{(n+1)!}.$$

   If this holds for every n and Taylor's Theorem is satisfied, then Taylor series converges to $f(x)$.

5. Example #1:
   (a) Determine the Taylor Series for $\sin x$ at $x = 0$.
   (b) Use the Remainder Estimation Theorem with $M = 1$ and $R_{2k+1}(x)|$ to see that the series converges for all x.
   (c) Find the Taylor series for $\sin 3x$ at $x = 0$.
   (d) Find the Taylor series for $x^2 \sin x$ at $x = 0$.
   (e) Calculate $\sin 1$ with an error of less than 10^{-6}.

6. Euler's Identity: $e^{i\theta} = \cos\theta + i\sin\theta$.

7. Example #2: Find the Taylor series at $x = 0$ for $x \ln(1 + 2x)$.

8. Example #3: The following series is the Maclaurin series of a function $f(x)$ at some point. Determine the function and the point, as well as the sum of the series.

$$1 - \frac{\pi^2}{4^2 \cdot 2!} + \frac{\pi^4}{4^4 \cdot 4!} - \cdots + \frac{(-1)^k \pi^{2k}}{4^{2k} \cdot (2k)!} + \cdots$$