# Section 5.4 - The Fundamental Theorem of Calculus

1. Mean Value Theorem:
   Suppose that $f$ is a function which is continuous on $[a,b]$ and differentiable on $(a,b)$. Then, there exist a point $c$ in $(a,b)$ for which

   $$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

   In other words, $f(b)-f(a) = f'(c)(b-a)$.

2. Fundamental Theorem of Calculus Part 1:
   Let $f$ be a continuous function on an interval $I$ and let $a$ be any number in $I$. Then if

   $$F(x) = \int_a^x f(t)\ dt,$$

   is differentiable on $I$, it follows that $F'(x) = f(x)$.

   **Proof:**
   $F(x) = \displaystyle\int_a^x f(t)\ dt$ and $F(x+h) = \displaystyle\int_a^{x+h} f(t)\ dt$

   $\Rightarrow F(x+h) - F(x) = \displaystyle\int_x^{x+h} f(t)\ dt.$

   By the Mean Value Theorem, there exists some $c$ in $(x, x+h)$ for which

   $$F'(c) = \frac{F(x+h)-F(x)}{h}.$$

   Hence, we have

   $$F'(c) = \frac{\int_x^{x+h} f(t)\ dt}{h}.$$

   If we let $h \to 0$, then $c \to x$, and we have

   $$F'(x) = \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\ dt}{h}.$$

   Since $f$ is continuous on $[x, x+h]$, there exist $m, M$ such that $m \leq f(t) \leq M$ for all $t$ in $[x, x+h]$. We hence know that

   $$mh \leq \int_x^{x+h} f(t)\ dt \leq Mh, \text{ or}$$

   $$m \leq \frac{\int_x^{x+h} f(t)\ dt}{h} \leq M$$

   If we let $h \to 0$, then $m = f(x) = M$, and we have

   $$f(x) \leq \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\ dt}{h} \leq f(x).$$

   It follows that

   $$F'(x) = \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\ dt}{h} = f(x).$$

3. Fundamental Theorem of Calculus Part 2:
   If $f$ is continuous on $[a, b]$, then

$$\int_a^b f(x)\ dx = F(b) - F(a),$$

   where $F$ is any antiderivative of $f$. $(F'(x) = f(x))$

   **Proof:**
   By the Fundamental Theorem Part 1, we have

$$g(x) = \int_a^b f(t)\ dt$$

   is an antiderivative of $f(x)$. It follows that $F(x) = g(x) + C$ for $a < x < b$. Thus,

$$F(b) = \int_a^b f(t)\ dt + C \qquad \text{and} \qquad F(a) = \int_a^a f(t)\ dt + C.$$

   It follows that

$$F(b) - F(a) = \int_a^b f(t)\ dt + C - \left(\int_a^a f(t)\ dt + C\right) = \int_a^b f(t)\ dt.$$

4. Example: Find the following derivatives:
   (a) $\dfrac{d}{dx}\displaystyle\int_0^x \cos(t^2)\ dt$
   (b) $\dfrac{d}{dx}\displaystyle\int_0^{x^2} \ln t\ dt$

5. Example: Let $F(x) = \displaystyle\int_0^x \sin(2t)\ dt$.
   (a) Evaluate $F(\pi)$.
   (b) Draw a sketch to explain geometrically why the answer to part (a) is correct.
   (c) For what values of $x$ is $F(x)$ positive? Negative?

6. Examples:
   (a) $\displaystyle\int_1^2 \frac{t^6 - t^2}{t^4}\ dt$
   (b) $\displaystyle\int_{-1}^1 \frac{3}{t^4}\ dt$ ( DNE, as not continuous on $[-1, 1]$).