Section 5.5 - Indefinite Integrals and the Substitution Rule

1. Elementary functions: combining constants, powers of x, $\sin x$, $\cos x$, e^x, $\ln x$, using addition, subtraction, multiplication, division, or composition of functions.

2. Guess and check: Ex. $f'(x) = e^{3x}$ or $g'(x) = \sin 5x$.

3. Chain rule: $\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x)$.

   (a) $\frac{d}{dx}\left(e^{x^3-5x}\right)$

   (b) $\frac{d}{dx}\left(\cos x^4\right)$

   (c) $\frac{d}{dx}\left(\ln(x^2+3x-1)\right)$

4. For antidifferentiating, we look for a product of two factors, where the "outside" factor is the derivative of the "inside" factor.

5. Substitution Method: Let u be the "inside function" and $du = u'(x)dx$.

   (a) Substitute

   (b) Integrate

   (c) Substitute back

6. Why does substitution work? (Can we write $\frac{du}{dx} \cdot dx = du$?)

   Observe that $\int f(g(x))g'(x)\, dx = F(g(x)) + C$.

   Let $u = g(x)$ and $\frac{du}{dx} = g'(x)$. Then

   $$\int f(u)\frac{du}{dx}\, dx = F(u) + C.$$

   But, since $F' = f$, we also know that

   $$\int f(u)\, du = F(u) + C$$

   Equating the two integrals, we get

   $$\int f(u)\frac{du}{dx}\, dx = \int f(u)\, du.$$

7. Examples: Evaluate the following integrals by substitution. (Observe constants are fine.)

   (a) $\int x^2(1+2x^3)^2\, dx$

   (b) $\int \frac{1}{\sqrt{3x-2}}\, dx$

8. More complex substitutions: ($u = 1 + x$ or polynomial division)
$$\int \frac{x}{1+x}\,dx$$

9. When to use substitution: (Undoes Chain Rule)

   (a) If you pick u, du also occurs.

   (b) Addition in the denominator and $du = k \cdot dx$

10. Example: Solve the initial value problem

$$\frac{dy}{dx} = 4x(x^2+8)^{-1/3}, \quad y(0) = 0.$$