Section 6.3 - Lengths of Plane Curves

1. Arc length of a parametric curve $(x = f(t), y = g(t))$, $a \leq t \leq b$.

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

By the Mean Value Theorem

$$\Delta x_k = f(t_k) - f(t_{k-1}) = f'(t_k^*)\Delta t_k, \qquad \Delta y_k = g(t_k) - g(t_{k-1}) = g'(t_k^{**})\Delta t_k$$

For a curve given parametrically, $a \leq t \leq b$, then

$$\text{Arc length of curve} = \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{[f'(t_k^*)]^2 + [g'(t_k^{**})]^2}\Delta t_k = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

Note: Arc length of a curve is the same for different parameterizations.

2. Example #1: Check for unit circle: Arc length = circumference.

3. Example #2: Find the length of the parametric curve given by $x = e^t \cos t$, $y = e^t \sin t$ for $0 \leq t \leq \pi$.

4. Arc length:
If $y = f(x)$, $a \leq x \leq b$, trivial parametrization is $x = t$, $y = f(t)$. Hence, $\dfrac{dx}{dt} = 1$, $\dfrac{dy}{dt} = f'(t)$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = f'(t)$.

$$\text{Arc length} = \sum \sqrt{1 + (f'(x))^2}\ \Delta x.$$

For $a < b$, the arc length of the curve $y = f(x)$ from $x = a$ to $x = b$ is given by

$$\text{Arc length} = \int_a^b \sqrt{1 + (f'(x))^2}\ dx.$$

5. Example #3: Find the arc length of the function $y = (3/4)x^{4/3} - (3/8)x^{2/3} + 5, \quad 1 \leq x \leq 8$.

6. Formula for the length of $x = g(y)$, $c \leq y \leq d$:

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy = \int_c^d \sqrt{1 + [g'(y)]^2}\ dy.$$

7. Example #4: (No easy antiderivative. Can find using graphing calculator.)
Find the length of the curve $x = \sin y$, $0 \leq y \leq \pi$.