Section 6.4 - Moments and Center of Mass

1. **Finding the total quantity from density:** Divide the region into small pieces in such a way that the density is approximately constant on each piece, and add the contributions of the pieces.

2. Example:
   A rod has length 2 meters. At a distance x meters from its left end, the density of the rod is given by
   $$\delta(x) = 2 + 6x \text{ g/m}.$$
   (a) Write a Riemann sum approximating the total mass of the rod.
   (b) Find the exact mass by converting the sum into an integral.

3. Center of Mass (Some SUV's tip over due to high centers of mass) - 3 dimensions = multiple integrals
   Balancing a teeter-totter

4. Two children with masses 25 kg and 30 kg are going to play on a seesaw that is 5 m long. Where should the fulcrum be placed so that the seesaw will balance?

5. Masses along a line with a fulcrum at the origin.
   - Each mass m_k exerts a downward force $m_k \cdot g$
   - Turning effect (torque) = $m_k \cdot g \cdot x_k$ (signed distance)
   - System torque = $m_1 g x_1 + m_2 g x_2 + m_3 g x_3 = g(m_1 x_1 + m_2 x_2 + m_3 x_3)$
   - The system will balance if and only if the torque is zero.
   - Moment of the system about the origin = $(m_1 x_1 + m_2 x_2 + m_3 x_3)$
   - Torque of m_k about $\bar{x} = (x_k - \bar{x}) m_k g$.
   - To balance, need $\sum (x_k - \bar{x}) m_k g = 0$ or $\bar{x} = \frac{\sum m_k x_k}{\sum m_k} = \frac{\text{moment}}{\text{mass}}$
     $\bar{x}$ is called the center of mass

6. **Center of mass** of a system of n point masses $m_1, m_2, \ldots, m_n$ located at positions $x_1, x_2, \ldots, x_n$ along the x-axis is given by
   $$\bar{x} = \frac{\sum_{i=1}^{n} x_i m_i}{\sum_{i=1}^{n} m_i} = \frac{\text{system moment about origin}}{\text{system mass}}.$$

7. Continuous (Variable) Mass Density:
   The **center of mass** $\bar{x}$ of an object lying along the x-axis between $x = a$ and $x = b$ is
   $$\bar{x} = \frac{\int_a^b x\delta(x)\, dx}{\int_a^b \delta(x)\, dx} = \frac{\text{moment about origin}}{\text{mass}},$$
   where $\delta(x)$ is the density (mass per unit length) of the object.

8. Example #1: Find the rod's moment about the origin, mass, and center of mass.

$$\delta(x) = 2 - (x/4), \quad 0 \le x \le 4.$$

9. Two- and three-dimensional regions:
For a region of constant density δ, the center of mass is given by ($A(x)$ = strip area)

$$\bar{x} = \frac{\int x\delta A_x(x)\ dx}{\text{Mass}} = \frac{M_y}{M}, \qquad \bar{y} = \frac{\int y\delta A_y(x)\ dy}{\text{Mass}} = \frac{M_x}{M}, \qquad \bar{z} = \frac{\int z\delta A_z(x)\ dz}{\text{Mass}}.$$

10. Example #2: Center of mass of a planar lamina (thin, flat plate of constant density)
Find the center of mass of the lamina of uniform density δ bounded by the graph of $f(x) = 4 - x^2$ and the x-axis. Ans. $(0, \frac{8}{5})$

11. Example #3: Find the center of mass of a thin plate covering the region bounded below by the parabola $y = x^2$ and above by the line $y = x$ if the plate's density at the point (x, y) is $\delta(x) = 12x$. (Use vertical strip approach)
Note: $\bar{x} = \frac{\int_0^1 x(x-x^2)\cdot 12x\ dx}{\int_0^1 (x-x^2)\cdot 12x\ dx}$, $\bar{y} = \frac{\int_0^1 \frac{(x+x^2)}{2}(x-x^2)\cdot 12x\ dx}{\int_0^1 (x-x^2)\cdot 12x\ dx}$
Ans. $(\frac{3}{5}, \frac{1}{2})$