Section 8.8 - Improper Integrals

1. Example #1: $\int_1^\infty \frac{dx}{x^2}$. (Type 1 Improper Integral)
   (Refer to page 614 for illustrations.)

2. If $\lim_{b\to\infty} \int_a^b f(x)\, dx$ is finite, then $\int_a^\infty f(x)\, dx$ **converges** and

$$\int_a^\infty f(x)\, dx = \lim_{b\to\infty} \int_a^b f(x)\, dx.$$

   Otherwise, $\int_a^\infty f(x)\, dx$ **diverges**. (Similar for $\int_{-\infty}^b f(x)\, dx$.)

3. Example #2: $\int_1^\infty \frac{dx}{x^3}$ and $\int_1^\infty \frac{dx}{x}$.
   Although both functions approach the x-axis, one does not do so fast enough.

4. Example #3: For what values of p does the integral $\int_1^\infty \frac{dx}{x^p}$ converge?

5. $\int_{-\infty}^\infty f(x)\, dx = \int_{-\infty}^c f(x)\, dx + \int_c^\infty f(x)\, dx$
   $\int_{-\infty}^\infty f(x)\, dx$ converges $\Leftrightarrow$ $\int_{-\infty}^c f(x)\, dx$ **and** $\int_c^\infty f(x)\, dx$ **both converge**.

6. Example #4: $\int_{-\infty}^\infty \frac{dx}{1+x^2}$.

7. Improper Integrals: Integrals with Infinite Discontinuities. (Type 2 Improper Integral)
   Example #5: $\int_0^1 \frac{dx}{\sqrt{1-x}}$.

8. If f is continuous on $[a, b)$ and $\lim_{x\to b} f(x) = \infty$, then

$$\int_a^b f(x)\, dx = \lim_{k\to b^-} \int_a^k f(x)\, dx.$$

   Define convergence and divergence.

9. Example #6: $\int_1^4 \frac{dx}{(x-2)^{2/3}}\, dx$ (Converges to $3 + \sqrt[3]{2}$)

10. If f is continuous on $[a, c) \cup (c, b]$ and $\lim_{x\to c} f(x) = \infty$, then

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

   Define convergence (both integrals converge) and divergence (at least one integral diverges).

11. Example #7: $\displaystyle\int_0^2 \frac{dx}{1-x}$ (Diverges)

12. Sometimes it's tough to determine the exact value of an improper integral, but it may be possible to compare a given integral to one who's behavior we already know.

13. Example #8: Determine whether $\displaystyle\int_1^\infty \frac{1}{\sqrt[3]{x^4+3}}\,dx$ converges.

14. Direct Comparison Test for $\displaystyle\int_a^\infty f(x)\,dx$:

    - If $0 \le f(x) \le g(x)$ and $\displaystyle\int_a^\infty g(x)\,dx$ converges, then $\displaystyle\int_a^\infty f(x)\,dx$ converges.
    - If $0 \le g(x) \le f(x)$ and $\displaystyle\int_a^\infty g(x)\,dx$ diverges, then $\displaystyle\int_a^\infty f(x)\,dx$ diverges.

15. Example #9: Determine whether $\displaystyle\int_0^\infty \frac{x^3+3x^2+4}{x^4+2x+11}\,dx$ converges.

16. Useful Integrals for Comparison:

    - $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges for $p > 1$ and diverges for $p \le 1$.
    - $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$ converges for $p < 1$ and diverges for $p \ge 1$.
    - $\displaystyle\int_0^\infty e^{-ax}\,dx$ converges for $a > 0$.

17. Limit Comparison Test: If the positive functions f and g are continuous on $[a, \infty)$ and if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)\,dx \qquad \text{and} \qquad \int_a^\infty g(x)\,dx$$

both converge or diverge.

18. Example #10: Decide if each improper integral converges or diverges. (Improper Handout)

    (a) $\displaystyle\int_0^1 \frac{d\theta}{\sqrt{\theta^3+\theta}}$ (Converges)

    (b) $\displaystyle\int_1^\infty \frac{\cos\phi}{\phi^2}\,d\phi$ (Converges)

    (c) $\displaystyle\int_0^\infty \frac{dz}{e^z+2^z}$ (Converges)