Section 2.3: Multiple Quantifiers

1. Ambiguous Language: Informal statements may be open to multiple interpretations. In such cases, we need to use context to determine meeting.

2. What does it mean to say that "A female student has the highest grade in every class I'm teaching?" In math/logic/cs, it's essential that all statements are interpreted the same way.

3. Multiple Quantifiers:
   $\forall\, x \in D,\, \exists\, y \in E$ s.t. x and x satisfy $P(x, y)$. You let someone else pick any $x \in D$, and you must find some $y \in E$ that "works" for that x.

4. Example #1: (p. 108, pr. 4-There is no biggest number)
   $\forall x \in \mathbb{R}, \exists\, n \in \mathbb{Z}$ s.t. $n > x$. Find an n for each x:
   (a) $x = 15.83$
   (b) $x = 10^8$
   (c) $x = 10^{10^{10}}$

5. Example #2:
   - There is a smallest positive integer: 1.
   - There is no smallest positive real number.
   - Definition of sequence limit: $\lim\limits_{n\to\infty} a_n = L$ means
     $\forall \epsilon > 0,\, \exists\, N \in \mathbb{Z}$ s.t. $n > N \Rightarrow a_n$ is between $L - \epsilon$ and $L + \epsilon$.

6. Negations of Multiply-Quantified Statements:
   - $\sim (\forall\, x \in D, \exists\, x \in E$ s.t. $P(x, y)) \equiv \exists\, x \in D$ s.t. $\forall\, y \in E, \sim P(x, y)$.
   - $\sim (\exists\, x \in D, \forall\, y \in E, P(x, y)) \equiv \forall\, x \in D,\ \exists\, y \in E$ s.t. $\sim P(x, y)$.

7. Example #3: Rewrite the following informally and write negations formally and informally.
   (a) $\exists$ a book b s.t. $\forall$ people p, p has read b.
   (b) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ s.t. $x + y = 0$.

8. Warning: If a statement consists of two different quantifiers, reversing their order can change truth value. (If the two quantifiers are of the same type, then reversing doesn't affect meaning.)

9. Example #4: Consider the following 2 statements:
   (a) $\forall$ homework assignments x, $\exists$ a person y s.t. y completed the assignment.
   (b) $\exists$ a person y s.t. $\forall$ homework assignments x, y completed x.

   Do they have the same meaning? Why or why not?

10. Example #5: Write a new statement by interchanging $\exists$ and $\forall$.
    Then, state which is true: either, neither, or both.
    (a) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ s.t. $x \cdot y = x$.
    (b) $\exists x \in \mathbb{R}^+,$ s.t. $\forall y \in \mathbb{R}^+,\ x \cdot y = 1$.

11. Example #6: $\exists! =$ there exists a unique.
    $\forall x \in \mathbb{R}^+,\ \exists!\ y \in \mathbb{R}^+$ s.t $\ y^2 = x$.