

## Section 3.1: Direct Proof I: Introduction

1. When you are first learning to write proofs, it is better to err on the side of supplying too many reasons rather than too few.
2. We first must assume certain axioms for real numbers: (Appendix A)
  - (a) Commutative Laws
  - (b) Associative Laws
  - (c) Distributive Laws (left and right)
  - (d) Existence of Identities
  - (e) Existence of Additive and Multiplicative Inverses (all but 0)
  - (f)  $A = A$ ; if  $A = B$  then  $B = A$ ; and if  $A = B$  and  $B = C$  then  $A = C$ .
3. Define **even** and **odd** integers.
  - (a) Is 0 even?
  - (b) Is -3 odd?
  - (c) If  $m$  and  $n$  are integers, is  $12m^2n^3$  even?
  - (d) Is every integer either even or odd?
  - (e) If  $m$  and  $n$  are integers, is  $6m + 8n + 3$  odd?
4. Define **prime** and **composite** positive integer.
  - (a) Is 1 prime?
  - (b) Is every integer greater than 1 either prime or composite?
5. Proving Existential Statement:  
Either find an  $x$  or give directions for finding an  $x$  s.t.  $Q(x)$  is true.
6. Example #1:  
Prove that there are integers  $m$  and  $n$  s.t.  $m > 1$  and  $n > 1$  and  $\frac{1}{m} + \frac{1}{n}$  is an integer.
7. Example #2: Disproving Universal Statements by Counterexample  
For all integers  $m$  and  $n$ , if  $2m + n$  is odd then  $m$  and  $n$  are both odd.
8. Proving Universal Statements:
  - Method of Exhaustion: Prove  $\forall n \in \mathbb{Z}$  s.t.  $1 \leq n \leq 10$ ,  $n^2 - n + 11$  is prime.
  - Generalizing from Generic Particular: Pick an arbitrary  $x$  so that results can be generalized to all other elements of the domain.
9. Direct Proof Method:
  - (a) Express the statement to be proved as  $\forall x \in D, P(x) \Rightarrow Q(x)$ . (Mentally)
  - (b) Suppose that  $x$  is an arbitrary element of  $D$  for which  $P(x)$  is true.
  - (c) Show  $Q(x)$  is true using definitions, previously established results, and rules for logical inference.

10. Example #3: Prove the following. (Q.E.D. = “quod erat demonstrandum” = “this is what we needed to show.”)

- (a) Prove that the sum of any two odd integers is odd.
- (b) For all integers  $n$ , if  $n$  is odd then  $n^2$  is odd.

11. Rules of style:

- (a) Copy the statement to be proved on your paper.
- (b) Clearly mark the beginning of your proof with the word “proof.”
- (c) Make your proof self-contained: Let  $m, n \in \mathbb{Z}$ .
- (d) Write your proof in complete sentences.
- (e) Give a reason for each assertion in your proof.
- (f) Include words like then, thus, so, hence, therefore, it follows that, . . .

Note: There are usually difference in peoples' proof styles and notation.

12. Common proof mistakes:

- Arguing from examples.
- Using the same letter to mean two different things.
- Jumping to a conclusion.
- Begging the question (circular reasoning).
- Misuse of the word *if*. (Don't use “if” when you mean “because”.)

13. Showing that an Existential Statement is False: Need to show negation (universal) is true.

14. Example #4: Show that there exists no  $n \in \mathbb{Z}^+$  s.t.  $4(n^2 + n + 1) - 3n^2$  is not a perfect square.

15. Conjecture, Proof, and Disproof:

- (a) Goldbach's Conjecture: Every even integer greater than 2 can be represented as the sum of two prime numbers. (Has been demonstrated up to  $10^{16}$ .)
- (b) Wiles proved Fermat's Last Theorem:  $\exists$  no  $x, y, z \in \mathbb{Z}^+$  s.t.  $x^n + y^n = z^n$  if  $n \geq 3$ .
- (c) Euler (1707-1783) conjectured that  $a^4 + b^4 + c^4 = d^4$  had no nontrivial whole number solution. In 1987, Noam Elkies proved it wrong:  $95,800^4 + 217,519^4 + 414,560^4 = 422,481^4$ .