Section 3.3: Direct Proof III: Divisibility

1. Number Theory: the study of properties of integers.

2. If n and d are integers, then n is **divisible** by d iff $n = dk$ for some $k \in \mathbb{Z}$. Alternatively,

   - n is a **multiple** of d
   - d is a **factor** of n
   - d is a **divisor** of n
   - d **divides** n
   - $d|n$ ($d|n \Leftrightarrow \exists\, k \in \mathbb{Z}$ s.t. $n = dk$.)

   Go through some examples (including negative numbers).

3. Show each of the following.

   - Divisors of zero.
   - If $a, b \in \mathbb{Z}^+$ and $a|b$, then $a \leq b$.
   - The only divisors of 1 are ± 1.
   - If a and b are integers, then $7a + 21b$ is divisible by 7.

4. $d \mid n \quad \Leftrightarrow \quad \exists\, k \in \mathbb{Z}$ s.t. $n = dk$.
   (Note negation.)

5. Prime numbers: $n \in \mathbb{Z}$, $n > 1$ is prime iff its only positive divisors are n and itself.

6. Theorem 3.3.1 (Transitivity of Divisibility)
   $\forall a, b, c \in \mathbb{Z}$, $a|b$ and $b|c \Rightarrow a|c$. (Prove this.)

7. Theorem 3.3.2 (Divisibility by a Prime)
   Any integer $n > 1$ is divisible by a prime number. (Prove this: induction-like.)

8. Example #1: Find a counterexample, alter, and prove.
   $\forall\, a, b \in \mathbb{Z}$, if $a|b$ and $b|a$ then $a = b$.

9. Example #2: True/False. Prove by definition/Counterexample.
   A sufficient condition for an integer to be divisible by 8 is that it be divisible by 16.

10. Fundamental Theorem of Arithmetic (Unique Factorization Theorem): Given $n \in \mathbb{Z}$ $(n > 1)$, $\exists\, k \in \mathbb{Z}^+$, distinct primes $p_1, p_2, \ldots, p_k$, and positive integers $e_1, e_2, \ldots, e_k$ s.t.

    $$n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}.$$

    This expression is unique, up to ordering. (**standard factored form** if primes increasing).