Section 3.4: Direct Proof IV: Division into Cases and QR-Theorem

1. Theorem 3.4.1 (The Quotient-Remainder Theorem)
   Given any integer n and positive integer d, there exist unique integers q and r s.t.
   $$n = dq + r \qquad \text{and } 0 \leq r < d.$$
2. Example #1: Pick Positive and Negative Integers and find q and r.
3. Introduce **div** (q) and **mod** (r). Now, n must be nonnegative, d positive.
4. Example #2: Determine days of the week for 2008 and 2009 (365 mod 7 = 1 and 366 mod 7 = 2).
5. Definition: Even/odd **parity**.
6. Theorem 3.4.2 Any two consecutive integers have opposite parity.
   (Prove by splitting into 2 cases: first integer is odd/even.)
7. Representations mod 3, 4, etc. - n mod 5 can be written in one five forms by Q-R Theorem:
   $$n = 5q, \qquad n = 5q+1, \qquad n = 5q+2, \qquad n = 5q+3, \qquad n = 5q+4.$$
8. Theorem 3.4.3 The square of any odd integer has the form $8m + 1$ for some integer m.
   pf.
   - Let n be any odd integer.
   - By QR-Theorem, n can be written as
     $$n = 4q, \qquad n = 4q+1, \qquad n = 4q+2, \qquad \text{or } n = 4q+3$$
     for some $q \in \mathbb{Z}$.
   - Since n is odd, we must have either $n = 4q + 1$ or $n = 4q + 3$.
   - Prove in cases.
9. Example #3: Prove that $n^2 - n + 3$ is odd, $\forall\, n \in \mathbb{Z}$.
10. Example #4: Use the QR-Theorem with $d = 3$ to prove that the product of any two consecutive integers has the form $3k$ or $3k + 2$ for some $k \in \mathbb{Z}$.