

Section 3.6: Indirect Argument: Contradiction and Contraposition

1. Method of Proof by Contradiction:

- (a) Suppose that the statement to be proved is false (i.e. the negation is true).
- (b) Show that this supposition leads logically to a contradiction.
- (c) Conclude that the statement to be proved is true.

2. Example #1: (p. 178, pr. 5, 7, 9, 11) Prove each statement by contradiction.

- (a) There is no greatest even integer.
- (b) There is no least positive rational number.
- (c) For all $x, y \in \mathbb{R}$, if x is irrational and y is rational, then $x - y$ is irrational.
- (d) If $a, b \in \mathbb{Q}$, $b \neq 0$, and r is an irrational number, then $a + br$ is irrational.

3. Theorem 3.6.2 There is no integer that is both even and odd. (Prove by contradiction)

4. Example #2: (p. 179, pr. 27)

Find the mistake in the following “proof” that every integer is rational.

Suppose not. Suppose every integer is irrational. Then the integer 1 is irrational. But $1 = 1/1$, which is rational. This is a contradiction, and hence the theorem is true.

5. Method of Proof by Contraposition:

- (a) Express the statement to be proved in the form $P(x) \Rightarrow Q(x)$.
($\forall x \in D$, if $P(x)$ then $Q(x)$.)
- (b) Rewrite this statement in contrapositive form: $\sim Q(x) \Rightarrow \sim P(x)$.
- (c) Prove the contrapositive by direct proof.
(Suppose $\exists x \in D$ s.t. $Q(x)$ is false. Show that $P(x)$ is false.)

6. Example #2: (p. 178, pr. 20, 22, 23) Prove each statement by contraposition.

- (a) For all real numbers r , if r^2 is irrational, then r is irrational.
- (b) The reciprocal of any irrational number is irrational.
- (c) For all integers n , if n^2 is odd, then n is odd.

7. Relation between proof by contradiction and proof by contraposition.

- Proof by contraposition: Suppose $\exists x \in D$ s.t. $\sim Q(x)$. Show $\sim P(x)$.
- Proof by contradiction: Suppose $\exists x \in D$ s.t. $P(x)$ and $\sim Q(x)$.
Find contradiction: $P(x)$ and $\sim P(x)$.
- When using contraposition, you know exactly what conclusion you need to show.
When using proof by contradiction, it's difficult to know what contradiction to head for.
- Advantage of contraposition: avoid finding the negation of a complicated statement.
- Disadvantage of contraposition: can only use for universal, conditional statements.