Section 4.4: Strong Induction and Well Ordering

Strong Induction:

1. Principle of Strong Mathematical Induction:
   Let $P(n)$ be defined for $n \in \mathbb{Z}$, and $a, b \in \mathbb{Z}$ be fixed. Then, if

   (a) $P(a), P(a+1), \ldots, P(b)$ are all true. (Basis step)

   (b) For all $k \in \mathbb{Z}, k > b$, $P(i)$ true for all $a \leq i < k \Rightarrow P(k)$ is true.

   Then $P(n)$ is true $\forall\ n \in \mathbb{Z}, n \geq a$.

2. The inductive hypothesis for strong induction assumes not only that $P(n)$ is true for a single value k (as with Mathematical Induction) but for **all** n up to, but not including, k.

3. Any statement that can be proved with MI can be proved with SMI. (Explain why.)
   The converse is also true but takes a little more work.

4. Example #1: (p. 242, pr. 2)
   Suppose $b_1, b_2, b_3, \ldots$ is a sequence defined as follows:

   $$b_1 = 4, \quad b_2 = 12, \quad b_k = b_{k-2} + b_{k-1},\ \forall\ k \geq 3.$$

   Prove that b_n is divisible by 4 for all $n \in \mathbb{Z}$, $n \geq 1$.

5. Theorem 4.4.1 (Existence and Uniqueness of Binary Integer Representations)
   Given any $n \in \mathbb{Z}^+$, n has a unique representation in the form

   $$n = c_r \cdot 2^4 + c_{r-1} \cdot 2^{r-1+\cdots+c_2 \cdot 2^2 + c_1 \cdot 2 + c_0},$$

   where r is a nonnegative integer, $c_r = 1$, and $c_j = 1$ or 0 for all $j = 0, 1, 2, \ldots, r-1$.
   pf. Strong induction and divide into cases (k even/odd)

6. Example #2: (p. 242, pr. 14)
   Use strong mathematical induction to prove that for any integer $n \geq 2$, if n is even, then any sum of n odd integers is even, and if n is odd, then any sum of n odd integers is odd.

Well Ordering:

7. Well-Ordering Principle for the Integers:
   Any set S containing one or more integers, all of which are greater than some $a \in \mathbb{Z}$ (note a may not be in S) has a least element.

8. It can be showed that MI, SMI, and Well-ordering principles are all equivalent

9. Example #1: (p. 243, pr. 20)
   The Archimedian property for the rational numbers states that for all $r \in \mathbb{Q}$, $\exists\ n \in \mathbb{Z}$ s.t. $n > r$. Prove this property.
   pf. Divide into cases: $r \leq 0$ (trivial, $n = 1$), $r = \frac{a}{b} > 0$: $r < n \Leftrightarrow a < nb$.
   $1 < 2 \Rightarrow a < 2a, 1 < b \Rightarrow 2a < 2ab = nb \Rightarrow a < 2a < nb$.

10. Example #2: (p. 243, pr. 21)
    Use the result of the Example #1 and the well-ordering principle for the integers to show that given any $r \in \mathbb{Q}$, $\exists\ m \in \mathbb{Z}$ s.t. $m \leq r < m + 1$.
    pf. Divide into cases: $r < 0$, $r = 0$, and $r > 0$.
    For $r > 0$, $r = \frac{a}{b}$, use Example #1.
    For $r < 0$, modify the result for the $r > 0$ case.

11. Quotient-Remainder Theorem (Existence Part)
    Given any $n \in \mathbb{Z}$ and any $d \in \mathbb{Z}^+$, $\exists\ q, r \in \mathbb{Z}$ s.t.

$$n = dq + r \qquad \text{and} \qquad 0 \leq r < d.$$