Section 5.2: Properties of Sets

1. Most basic method used for proofs involving sets: element arguments
   Let X and Y be given. To prove $X \subseteq Y$,

   (a) Suppose that x is an arbitrarily chosen element of X,

   (b) show that $x \in Y$.

   ($X \subseteq Y \Leftrightarrow \forall\, x, x \in X \Rightarrow x \in Y$.)

2. Definitions of Set Operations:

   (a) $x \in X \cup Y \quad \Leftrightarrow \quad x \in X$ or $x \in Y$

   (b) $x \in X \cap Y \quad \Leftrightarrow \quad x \in X$ and $x \in Y$

   (c) $x \in X - Y \quad \Leftrightarrow \quad x \in X$ and $x \notin Y$

   (d) $x \in X^c \quad \Leftrightarrow \quad x \notin X$

   (e) $(x, y) \in X \times Y \quad \Leftrightarrow \quad x \in X$ and $y \in Y$

3. Theorem 5.2.1: Subset Relations

   (a) For all sets A and B, $A \cap B \subseteq A$ and $A \cap B \subseteq B$.

   (b) For all sets A and B, $A \subseteq A \cup B$ and $B \subseteq A \cup B$.

   (c) For all sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

   Prove this using an Element Argument and note relation to formal logic (specialization, etc.)

4. Set Identities: For all sets $A, B, C \subseteq U$

1. Commutative Laws:	$A \cup B = B \cup A$	$A \cap B = B \cap A$
2. Associative Laws:	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
3. Distributive Laws:	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
4. Identity Laws:	$A \cup \varnothing = A$	$A \cap U = A$
5. Complement Laws:	$A \cup A^c = U$	$A \cap A^c = \varnothing$
6. Double Complement:	$(A^c)^c = A$	
7. Idempotent Laws	$A \cup A = A$	$A \cap A = A$
8. Universal Bound Laws:	$A \cup U = U$	$A \cap \varnothing = \varnothing$
9. De Morgan's Laws:	$(A \cup B)^c = A^c \cap B^c$	$(A \cap B)^c = A^c \cup B^c$
10. Complements:	$U^c = \varnothing$	$\varnothing^c = U$
11. Set Difference Law:	$A - B = A \cap B^c$	

5. Proving Set Identities: Two sets are equal $\Leftrightarrow$ each is a subset of the other.

6. Method for Proving that Sets are Equal:
   Let sets X and Y be given. To prove that $X = Y$:

   (a) Prove that $X \subseteq Y$.

   (b) Prove that $Y \subseteq X$.

   Prove some of the above laws (Definitely DeMorgan's Law).

7. Theorem 5.2.4: If E is a set with no elements and A is any set, then $E \subseteq A$.
   Prove by contradiction.

8. Corollary 5.2.5 (Uniqueness of the Empty Set)
   There is only one set with no elements.
   Prove by contradiction, with Theorem 5.2.4.

9. Proposition 5.2.6:
   For all sets A, B, and C, if $A \subseteq B$ and $B \subseteq C^c$, then $A \cap C = \varnothing$.
   Prove by contradiction.