Section 7.2: One-to-one and Onto

1. A function $f : X \to Y$ is **one-to-one** (injective) $\Leftrightarrow$

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2, \qquad \forall\ x_1, x_2 \in X.$$

   Also give contrapositive, and what occurs if f is not one-to-one.

2. Go through one-to-one functions on finite (diagrams) and infinite (formulas) sets.

3. Showing that a function is one-to-one: (Go through examples of each.)

   (a) Suppose $x_1, x_2 \in X$ with $f(x_1) = f(x_2)$.
   (b) Show that $x_1 = x_2$.

   Showing that a function is not one-to-one:

   - Find $x_1, x_2 \in X$ s.t. $x_1 \neq x_2$ but $f(x_1) = f(x_2)$.

4. A function $f : X \to Y$ is **onto** (surjective) $\Leftrightarrow$

$$\forall\ y \in Y,\ \exists\ x \in X \text{ s.t. } f(x) = y.$$

   Also, give negation (not onto).

5. Go through onto functions on finite (diagrams) and infinite (formulas) sets.

6. Showing that a function is onto: (Go through examples of each.)

   (a) Suppose y is any element in Y.
   (b) Show that $\exists\ x \in X$ s.t. $f(x) = y$.

   Showing that a function is not one-to-one:

   - Find an element $y \in Y$ s.t. $y \neq f(x)$ for all $x \in X$.

   (Go through scratch work and proof.)

7. A function f is a **bijection** if f is both one-to-one and onto.

8. Theorem: Suppose $f : A \to B$ and $g : B \to A$. (Prove)

   (a) If f and g are 1-1 functions, then $g \circ f$ is a 1-1 function.
   (b) If f and g are onto functions, then $g \circ f$ is an onto function.

9. Inverse functions:

   - We need $f \circ f^{-1}$ to be the **identity function** on $\mathcal{D}(f^{-1})$ and $f^{-1} \circ f$ to be the identity function on $\mathcal{D}(f)$.
   - Let $f : X \to Y$ be a 1-1 function. Then $f^{-1} : \mathcal{R}(f) \to X$ is the **inverse function** of f defined by $f^{-1}(y) = x$ if $y = f(x)$.
   - Finding f^{-1}.
   - If we view f as a series of ordered pairs, f^{-1} reverses coordinates. For f^{-1} to be a function, f must be 1-1.
   - $f^{-1}(B)$ is called the **inverse image** or **preimage** of B.

10. Example: Let $f : X \to Y$ with $A_1, A_2 \subset X$. Show that $f(A_1 \cup A_2) \subset f(A_1) \cup f(A_2)$.