Section 1.6 - Equilibria and the Phase Line

1. Redundancy in the slope field of an autonomous differential equation. The slope marks are parallel along horizontal lines in the ty-plane. If we know the slope field along a single vertical line $t = t_0$, then we know the slope field in the entire ty-plane. Hence, instead of drawing the entire slope field, we can draw just one line (**phase line**).

2. Example: Sketch the phase line for the differential equation

$$\frac{dy}{dt} = y^2 - 6y - 16.$$

   (Explain what this Existence, Uniqueness Theorems say about initial-value problem w/ $y(0) = 4$) Also, use phase lines to sketch solutions.

3. Graphs associated with $\frac{dy}{dt} = f(y)$

   - Graph of $f(y)$
   - Slope field
   - Graph of solutions ($y(t)$)
   - Phase line

4. How to Draw a Phase Line
   (Note that information about speed of increase and decrease is lost)

   (a) Draw the y-line
   (b) Find the equilibrium points, and mark them on the line.
   (c) Find the intervals for which $f(y) > 0$, and draw arrows pointing up in these intervals.
   (d) Find the intervals for which $f(y) < 0$, and draw arrows pointing down in these intervals.

5. Suppose that $y(t)$ is a solution to an autonomous differential equation

$$\frac{dy}{dy} = f(y).$$

   - If $f(y(0)) = 0$, then $y(0)$ is an equilibrium point and $y(t) = y(0)$ for all t.
   - If $f(y(0)) > 0$, then $y(t)$ is increasing for all t and either $y(t) \to \infty$ or y tends to the first equilibrium point larger than $y(0)$.
   - If $f(y(0)) < 0$, then $y(t)$ is decreasing for all t and either $y(t) \to -\infty$ or y tends to the first equilibrium point smaller than $y(0)$.

   Similar results hold as t decreases (as time runs backward).

6. Warning: Not All Solutions Exist for All Time

   - $\frac{dy}{dt} = (1 + y)^2$: Equilibrium point at y = -1. Analytically, look at solutions for which $y(0) > -1$. (They're unbounded in finite time) $\Rightarrow c < 0 \Rightarrow$ defined only for $t < -c$

- $\frac{dy}{dt} = \frac{1}{1-y}$: If $y = 1$, then dy/dt does not exist. The phase line has a hole in it. (Denoted by a small empty circle in the phase line) Once a solution reaches $y = 1$, it can't be continued because it has left the domain of definition of the differential equation.

7. Observe that we can draw a phase line from qualitative information: the graph of $f(y)$.

8. Illustrate the following with phase lines and solution pictures:

   - Sink: an equilibrium point y_0 for which any solution with initial condition sufficiently close to y_0 is asymptotic to y_0 as t increases.
   - Source: all solutions that start sufficiently close to y_0 tend toward y_0 as t increases.
   - Node: an equilibrium point that is neither a source nor a sink

9. Example: Sketch the phase line for the differential equation

$$\frac{dw}{dt} = w \cos w.$$

   Identify the equilibrium points as sinks, sources, or nodes.

10. Linearization Theorem
   Suppose y_0 is an equilibrium point of the differential equation $dy/dt = f(y)$ where f is a continuously differentiable function. Then,

   - if $f'(y_0) < 0$, then y_0 is a sink;
   - if $f'(y_0) > 0$, then y_0 is a source;
   - if $f'(y_0) = 0$, then we need additional information to determine the type of y_0.