Section 1.8 - Linear Differential Equations

1. **First-order Linear Differential Equation:**

$$\frac{dy}{dt} = g(t)y + r(t),$$

where $g(t)$ and $r(t)$ are arbitrary functions of t. (Dependent variable (y) only to 1^{st} power.) (Sometimes we must do a little algebra to see that an equation is linear.)

2. Examples of First-order Linear Differential Equations:

   (a) $\frac{dy}{dt} = t^3 y + t \sin t^2$

   (b) $\frac{dy}{dt} = \frac{e^{\sin t}}{t^3 - t^2} y + \ln t - 3t^2 + 1$

   (c) $yt + 7 = \frac{dy}{dt} + 4y$

   (Observe that some differential equations can be both linear and autonomous.)

3. Additional terminology:

   - A linear equation is **homogeneous** (*unforced*) if $b(t) = 0$ for all t.
   - Otherwise it is **nonhomogeneous** (*forced*).
   - A first-order linear differential equation is a *constant-coefficient* equation if the coefficient of y ($a(t)$) is constant.

4. Linear differential equations are used to mode decay of radioactive elements, cooling of beverages, mixing of chemicals in a solution, etc.

5. Example: $\frac{dy}{dt} = -4y + 3e^{-t}$

6. **Linearity Principle:** If $y_h(t)$ is a solution of the homogeneous linear equation

$$\frac{dy}{dt} = a(t)y,$$

then $ky_h(t)$ is also a solution for any constant k. (Prove this.)
Not surprising since it's separable.

7. **Extended Linearity Principle:** Consider the nonhomogeneous equation

$$\frac{dy}{dt} = a(t)y + b(t)$$

and its associated homogeneous equation

$$\frac{dy}{dt} = a(t)y.$$

(a) If $y_h(t)$ is any solution of the homogeneous equation and $y_p(t)$ is any solution of the nonhomogeneous equation , then $y_h(t) + y_p(t)$ is also a solution of the nonhomogeneous equation.

(b) Suppose $y_p(t)$ and $y_q(t)$ are two solutions of the nonhomogeneous equation. Then $y_p(t) - y_q(t)$ is a solution of the associated homogeneous equation.

Thus, if $y_h(t)$ is nonzero, $ky_h(t) + y_p(t)$ is the general solution of the nonhomogeneous equation.

8. Steps for solving a linear equation:

(a) Find the general solution to the homogeneous equation (separation of variables).

(b) Find one particular solution of the nonhomogeneous equation (may be difficult if $a(t)$ is not constant). Often done by guess and check.

(c) Add the two above solutions together to get the general nonhomogeneous solution.

9. Other examples:

(a) $\frac{dy}{dt} = 2y + \sin 2t$ (Guess $y_p(t) = \alpha \cos 2t + \beta \sin 2t$.)

(b) $\frac{dy}{dt} = \frac{y}{2} + 4e^{t/2}$ (Second guess: $y_p(t) = \alpha t e^{t/2}$).