# Section 1.9 - Integrating Factors for Linear Equations

1. Integrating Factors: $\frac{dy}{dt} = g(t)y + r(t)$

    - First, rewrite as $\frac{dy}{dt} + a(t)y = r(t)$.
    - Multiply through by some function $\mu(t)$ to get $\mu(t)\frac{dy}{dt} + \mu(t)a(t)y = \mu(t)r(t)$.
    - Now, assume that we have some function $\mu(t)$ for which

    $$\frac{d(\mu(t) \cdot y(t))}{dt} = \mu(t)\frac{dy}{dt} + \mu(t)a(t)y.$$

    - Then, the differential equation becomes $\frac{d(\mu(t) \cdot y(t))}{dt} = \mu(t)r(t)$.
    - Integrate both sides w.r.t. $t$: $\mu(t)y(t) = \int \mu(t)r(t)dt$, or

    $$y(t) = \frac{1}{\mu(t)} \int \mu(t)r(t)dt.$$

    - Product rule says: $\frac{\mu(t) \cdot y(t)}{dt} = \mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}y(t)$.
    - Hence, we must find $\mu(t)$ for which $\frac{d\mu}{dt}y(t) = \mu(t)a(t)y, or$

    $$\frac{d\mu}{dt} = \mu(t)a(t).$$

    - This gives us a new differential equation, which is separable:

    $$\frac{1}{\mu}\frac{d\mu}{dt} = a(t).$$

    - We hence have the solution $\mu(t) = e^{\int a(t)dt}$. **(Integrating factor)**

2. Example: Find the general solution.

$$\frac{dy}{dt} = \frac{3}{t}y + t^5$$

3. Example: Find the particular solution with the given initial condition.

$$\frac{dy}{dt} = -y + e^t, \qquad y(0) = 0.4$$

4. Example: Problems with integration
(Express with as few integrals as possible.)

$$\frac{dy}{dt} = 2ty + 1$$