Section 4.3 - Undamped Forcing and Resonance

1. Undamped Harmonic Oscillator:
$$m\frac{d^2y}{dt^2} + ky = 0$$
We'll deal only with $m = 1$.

2. Complex Eigenvalues
   - Consider the second-order differential equation $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 13y = 0$.
   - Eigenvalues are $-2 \pm 3i$.
   - Complex solution is $y(t) = e^{(-2+3i)t} = e^{-2t}(\cos 3t + i \sin 3t)$
   - We have two real solutions: $y_1(t) = e^{-2t}\cos 3t$ and $y_2(t) = e^{-2t}\sin 3t$.
   - The general solution is: $y(t) = k_1 e^{-2t}\cos 3t + k_2 e^{-2t}\sin 3t$.

3. Consider a particular harmonic oscillator equation with sinusoidal forcing given by
$$\frac{d^2y}{dt^2} + 2y = \cos \omega t.$$
The parameter ω controls the frequency of the external forcing. (Frequency $= \frac{1}{\text{period}}$.)
Forcing frequency $= \omega/2\pi$.
   - General solution to the unforced equation is $y_h(t) = k_1 \cos\sqrt{2}t + k_2 \sin\sqrt{2}t$.
     Natural frequency $= \sqrt{2}/(2\pi)$.
   - Use the Method of Undetermined Coefficients to find one solution of the forced equation.
$$y_c(t) = \frac{1}{2-\omega^2}e^{i\omega t} = \frac{1}{2-\omega^2}(\cos\omega t + i \sin \omega t)$$
$$y_p(t) = \frac{1}{2-\omega^2}\cos\omega t$$
   - General Solution:
$$y(t) = k_1\cos\sqrt{2}t + k_2 \sin\sqrt{2}t + \frac{1}{2-\omega^2}\cos\omega t$$
     Observe that $y(t)$ is undefined at $\omega = \sqrt{2} \approx 1.4$.

4. Qualitative Analysis: (Look at in HPGSystemSolver as ω varies from 0.5 to 1.2)
   - **Beating:** occurs when the natural response and forced response have approximately the same frequency.
   - Frequency of the beats $= \frac{\text{natural frequency} - \text{forced frequency}}{2}$. (Discussion in text.)
   - Frequency of rapid oscillations $= \frac{\text{natural frequency} + \text{forced frequency}}{2}$.
   - Determine these frequencies for our example with $\omega = 1.1$. (Look at in HPGSystemSolver.)
     Period of beats ≈ 40; Period of rapid oscillation ≈ 5.

5. Resonance:

- Forcing whose frequency is the same as the natural frequency of the oscillator is called **resonant forcing**, and the oscillator is said to be in **resonance**.
- Look at what happens in the general solution of our example as $\omega \to \sqrt{2}$.
- Compute the general solution in the resonant case:

$$\frac{d^2y}{dt^2} + 2y = \cos\sqrt{2}t.$$

  General solution: $y(t) = k_1 \cos\sqrt{2}t + k_2 \sin\sqrt{2}t + \dfrac{1}{2\sqrt{2}}\, t \sin\sqrt{2}t.$

- Point out Millennium Bridge example and Tacoma Narrows Bridge examples.