# Review of Chapter 3 Material

1. Consider an arbitrary linear system $\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$, where $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
    - The **characteristic polynomial** of the system is the polynomial
    $$\det(\mathbf{A} - \lambda \mathbf{I}) = \det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = (a - \lambda)(d - \lambda) - bc = 0.$$
    - We say that $\lambda$ is an **eigenvector** of a matrix $\mathbf{A}$ with corresponding **eigenvector** $\mathbf{V}$ if
    $$\mathbf{AV} = \lambda \mathbf{V}.$$
    - We observe that if $\lambda$ is an eigenvector of $\mathbf{A}$ with eigenvector $\mathbf{V}$, then
    $$\mathbf{Y}(t) = e^{\lambda t}\mathbf{V}$$
    is a solution to the system $\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$.
2. Theorem:
Suppose $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are solutions of the linear system $d\mathbf{Y}/dt = \mathbf{AY}$. If $\mathbf{Y}_1(0)$ and $\mathbf{Y}_2(0)$ are linearly independent, then for any initial condition $\mathbf{Y}(0) = (x_0, y_0)$, we can find constants $k_1$ and $k_2$ so that $k_1\mathbf{Y}_1(t) + k_2\mathbf{Y}_2(t)$ is the solution to the initial-value problem
$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}, \qquad \mathbf{Y}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$
It follows that $\mathbf{Y}(t) = k_1\mathbf{Y}_1(t) + k_2\mathbf{Y}_2(t)$ is the **general solution** to the system.
3. We now consider the various possibilities for eigenvalues of $\mathbf{A}$.
    - If $\mathbf{A}$ has two distinct, real eigenvalues, $\lambda_1$ and $\lambda_2$, with corresponding eigenvectors $\mathbf{Y}_1$ and $\mathbf{Y}_2$, respectively, then
    $$\mathbf{Y}(t) = k_1 e^{\lambda_1 t}\mathbf{V}_1 + k_2 e^{\lambda_2 t}\mathbf{V}_2$$
    is the general solution of the system. Furthermore
        - Any linear system with one positive eigenvalue and one negative eigenvalue acts as a source in the direction of one eigenvector and as a sink in the direction of the other. An equilibrium point of this form is called a **saddle**.
        - If $\lambda_1 < \lambda_2 < 0$ (both eigenvalues are negative), then the origin is a sink, and most solutions $\rightarrow (0, 0)$ tangent to eigenvectors for $\lambda_2$.
        - If $0 < \lambda_2 < \lambda_1$ (both eigenvalues are positive), then the origin is a source, and most solution curves leave the origin in the direction of the $\lambda_2$-eigenvectors.
        - If one of the eigenvectors is zero (assume $\lambda_1$=0), then
            - ∗ The general solution becomes $\mathbf{Y}(t) = k_1\mathbf{V}_1 + k_2 e^{\lambda_2 t}\mathbf{V}_2$.
            - ∗ Every point on the line of eigenvectors for $\lambda_1 = 0$ is an equilibrium point ($k_2 = 0$).
            - ∗ If $\lambda_2 < 0$, the solution tends to an equilibrium point $k_1\mathbf{V}_1$ along a line parallel to $\mathbf{V}_2$.
            - ∗ If $\lambda_2 > 0$, the solution moves away from the line of equilibrium points as $t \rightarrow \infty$.

- If $\mathbf{A}$ has two complex eigenvalues $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$, with eigenvectors $\mathbf{V}_1$ and $\mathbf{V}_2$ respectively, then

  $$\begin{aligned}\mathbf{Y}(t) &= e^{\lambda t}\mathbf{V}_1 \\ &= e^{(\alpha+i\beta)t}\mathbf{V}_1 \\ &= e^{\alpha t}e^{i\beta t}\mathbf{V}_1 \\ &= e^{\alpha t}(\cos\beta t + i\sin\beta t)\mathbf{V}_1\end{aligned}$$

  is a solution. We can rewrite $\mathbf{Y}(t)$ as

  $$\mathbf{Y}(t) = \mathbf{Y}_{\text{re}}(t) + i\mathbf{Y}_{\text{im}}(t).$$

  Furthermore, $\mathbf{Y}_{\text{re}}(0)$ and $\mathbf{Y}_{\text{im}}(0)$ are linearly independent, and hence the general solution is

  $$\mathbf{Y}(t) = k_1\mathbf{Y}_{\text{re}}(t) + k_2\mathbf{Y}_{\text{im}}(t).$$

  Qualitatively, if $\beta > 0$, then the solution curves spiral around the origin in the phase plane with **natural period** $2\pi/\beta$.

  - If $\alpha < 0$, then solutions spiral toward origin. (**spiral sink**)
  - If $\alpha > 0$, then solutions spiral away from origin. (**spiral source**)
  - If $\alpha = 0$, then solutions are periodic, same closed curve. (orgin=**center**)

- If $\mathbf{A}$ has a repeated real eigenvalue $\lambda$, then there only one line of eigenvectors, and the general solution has the form

  $$\mathbf{Y}(t) = e^{\lambda t}\mathbf{V}_0 + te^{\lambda t}\mathbf{V}_1,$$

  where $\mathbf{V}_0 = (x_0, y_0)$ is an arbitrary initial condition and $\mathbf{V}_1$ is determined from $\mathbf{V}_0$ by

  $$\mathbf{V}_1 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{V}_0.$$

  If $\mathbf{V}_1$ is zero, then $\mathbf{V}_0$ is an eigenvector and $\mathbf{Y}(t)$ is a straight-line solution. Otherwise, $\mathbf{V}_1$ is an eigenvector.
  (Warning: This doesn't say that $e^{\lambda t}\mathbf{V}_0$ and $te^{\lambda t}\mathbf{V}_1$ are solutions.)
  Qualitatively, we can summarize this case as follows:

  - If $\lambda < 0$, then as $t \to \infty$, $\mathbf{Y}(t) \to (0, 0)$ (sink) along $\mathbf{V}_1$ since $t\mathbf{V}_1$ dominates.
  - If $\lambda > 0$, then $\mathbf{Y}(t) \to \infty$ along $\mathbf{V}_1$ as $t \to \infty$.
  - We can think of the repeated eigenvalue case as a bifurcation between the 2 distinct, real eigenvalue case (2 straight-line solutions) and complex conjugate eigenvalue case (no straight-line solutions).

4. Constructing a Matrix with Desired Eigenvalues:
   Assume that you want to construct a matrix $\mathbf{A}$ with eigenvalues $\lambda_1$ and $\lambda_2$. We can do so by picking $\lambda_1$ and $\lambda_2$ to be the entries on the main diagonal of $\mathbf{A}$ and picking at least one of the other two entries to be zero. For example, consider

   $$\mathbf{A} = \begin{pmatrix} \lambda_1 & b \\ 0 & \lambda_2 \end{pmatrix}.$$

   You should quickly be able to verify that the eigenvalues of this matrix are $\lambda_1$ and $\lambda_2$.