Section 1.4 - Numerical Technique: Euler's Method

1. Initial Value Problem: $\frac{dy}{dt} = f(t, y)$, $y(t_0) = y_0$. (Illustrate this with a diagram)
   (a) Choose a small step size (Δt)
   (b) Take a step with t-axis size Δt and with direction determined by the slope field at (t_0, y_0). We arrive at (t_1, y_1), where $t_1 = t_0 + \Delta t$.
   (c) Repeat the procedure using (t_k, y_k) to find (t_{k+1}, y_{k+1}).

2. Euler's Method for $\frac{dy}{dt} = f(t, y)$.
   (Illustrate this with $\frac{dy}{dt} = t - y^2$, $y(0) = 2$, $0 \leq t \leq 2$, $\Delta t = 0.5$)
   Given $y(t_0) = y_0$ and step size Δt, compute (t_{k+1}, y_{k+1}) from (t_k, y_k) as follows:
   (a) Use the differential equation to compute the slope $f(t_k, y_k)$.
   (b) Calculate (t_{k+1}, y_{k+1}) using the formulas
   $$t_{k+1} = t_k + \Delta t, \text{ and}$$
   $$y_{k+1} = y_k + f(t_k, y_k)\Delta t.$$

3. Illustrate the above example using the EulersMethod, and compare the result to the known solution. Observe that decreasing the step size will increase the accuracy but also the amount of work required.

4. Nonautonomous Example:
   Perform Euler's method with $\Delta t = 0.25$ over the interval $0 \leq t \leq 1$ for the differential equation
   $$\frac{dy}{dt} = -ty$$
   and with initial condition $y(0) = 1$.

5. Refer to the textbook (p.62) for a discussion of Errors in Numerical Methods with
   $$\frac{dy}{dt} = e^t \sin y.$$
   Observe equilibrium solutions at $y(t) = n\pi$.
   Use $y(0) = 5$. As $t \to 5$, the solution tends toward the equilibrium solution $y(t) = \pi$. Just before $t = 5$, however, the approximation behaves erratically. The problem is that as t increases, the slopes in the slope field are large for large t. Even a small step in the t-direction throws us far from the actual solution.

   Moral: Sometimes numerical methods work beautifully; sometimes they fail.