Section 1.5 - Existence and Uniqueness of Solutions

1. How do we know if solutions exist?
   (Parallel to Intermediate Value Theorem-Tells us a solution exists but doesn't find it.)

2. Existence Theorem:
   Suppose $f(t, y)$ is a continuous function in a rectangle of the form $\{(t, y) : a < t < b, c < y < d\}$ in the ty-plane. If (t_0, y_0) is a point in this rectangle, then there exists an $\epsilon > 0$ and a function $y(t)$ defined for $t_0 - \epsilon < t < t + \epsilon$ that solves the initial-value problem
   $$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0.$$
   (In other words, as long as the right-hand side is reasonable, solutions exist.)

3. Existence Theorem only guarantees solutions in a small domain.
   Example: $\frac{dy}{dt} = 1 + y^2$, $y(0) = 0$. (Look at slope field in HPGSolver)
   Analytically, we obtain the general solution
   $$y(t) = \tan(t + c)$$
   and particular solution
   $$y(t) = \tan(t)$$
   on the domain $-\pi/2 < t < \pi/2$. (Blows up near vertical asymptotes)

4. Uniqueness Theorem: (Valuable theoretically and practically)
   Suppose $f(t, y)$ and $\partial f/\partial y$ are continuous functions in a rectangle of the form
   $$\{(t, y) : a < t < b, c < y < d\}$$
   in the ty-plane. If (t_0, y_0) is a point in this rectangle and if $y_1(t)$ and $y_2(t)$ are two functions that solve the initial-value problem
   $$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0$$
   for all t in the interval $t_0 - \epsilon < t < t + \epsilon$ ($\epsilon > 0$), then
   $$y_1(t) = y_2(t)$$
   for $t_0 - \epsilon < t < t + \epsilon$.
   (In other words, the solution is unique in this interval.)

5. Example where uniqueness fails:
   $$\frac{dy}{dt} = \sqrt{y}$$
   The Uniqueness Theorem tells us nothing about the number of solutions to an initial-value problem for which $y(t_0) = 0$, as $\frac{\partial f}{\partial y}$ doesn't exist when $y = 0$.
   (Solve analytically and observe equilibrium solution $y(t) = 0$ when initial-value is $y(0) = 0$.)

6. Uniqueness Theorem Restated:
   "If two solutions are ever in the same place at the same time, then they are the same function."

7. Uniqueness and Qualitative Analysis:
   Example: $\dfrac{dy}{dt} = f(t, y)$
   Assume that f satisfies the conditions of the Uniqueness Theorem in the entire ty-plane. Also, assume that the following are two solutions to the differential equation:

   $$y_1(t) = -1 \text{ for all } t, \qquad \text{and} \qquad y_2(t) = 1 + t^2 \text{ for all } t$$

   If we add in the additional condition $y(0) = 0$, what can we conclude about the solution to the initial-value problem?

8. Consider the differential equation

   $$\frac{dy}{dt} = (y-2)(y-3)y$$

   with initial condition $y(0) = 3$. What do the Existence and Uniqueness Theorems say about the corresponding solution?