Section 1.7 - Bifurcations

1. Linearization Theorem (Recall from Section 1.6)
   Suppose y_0 is an equilibrium point of the differential equation $dy/dt = f(y)$ where f is a continuously differentiable function. Then,
   - if $f'(y_0) < 0$, then y_0 is a sink;
   - if $f'(y_0) > 0$, then y_0 is a source;
   - if $f'(y_0) = 0$, then we need additional information to determine the type of y_0.
2. **Bifurcation:** A small change in the parameter that leads to a drastic change in long-term behavior.
3. Example: $\dfrac{dy}{dt} = y^2 + 3y + \mu$ (Depends on the parameter μ)
   We can write $f_\mu(y) = y^2 + 3y + \mu$, then, for instance $f_5(y) = y^2 + 3y + 5$.
   $$\frac{dy}{dt} = f_\mu(y).$$
   Take derivative, set =0, solve for y, then find μ. **One-parameter family** of differential equations.
   Check $\mu = -4$, $\mu = 0$, $\mu = 2$, $\mu = 3$. (μ need not be an integer.) Bifurcation value at 9/4.
   For all μ, draw the phase lines and solutions. (No arrows on phase line)
   Bifurcation occurs between $\mu = 2$ and $\mu = 3$.

4. **Bifurcation Diagram**
   A picture in the μy-plane of the phase lines near a bifurcation value. (See previous example)
   Phase lines are plotted for each value of μ, and equilibrium points are then plotted.
5. Example: A bifurcation from one to three equilibria
   $$\frac{dy}{dt} = g_\alpha(y) = y^3 + \alpha y = y(y^2 + \alpha).$$
   ($\alpha = 0$ is the bifurcation value.)
6. Bifurcations of Equilibrium Points:
   (For rest of lecture, assume $\dfrac{dy}{dt} = f_\mu(y)$, where $\dfrac{\partial f_\mu}{\partial y}$ and $\dfrac{\partial f_\mu}{\partial \mu}$ exist and are continuous.)
7. When Bifurcations do not happen:
   If y_0 is a sink with $f'_{\mu_0}(y_0) < 0$, then for all μ_1 sufficiently close to μ_0, there is a sink at some y_1 very near y_0.
   If y_0 is a source with $f'_{\mu_0}(y_0) > 0$, then for all μ_1 sufficiently close to μ_0, there is a source at some y_1 very near y_0.
   It follows that there are no bifurcations at these points.
   Hence, we look for values $\mu = \mu_0$ and $y = y_0$ for which $f_{\mu_0}(y_0) = 0$ and $f'_{\mu_0}(y_0) = 0$.

8. Example: A one-parameter family with two bifurcation points.

$$\frac{dy}{dt} = f_\mu(y) = y(1-y)^2 + \mu$$

(Bifurcations at $\mu = 0$ and $\mu = -4/27$: Set $f'_\mu(y) = 0$, then look at $y = 1/3$.)