Section 2.2 - The Geometry of Systems

1. Autonomous system: a system in which the independent variable does not occur on the right-hand side

2. Vector notation: A vector can be visualized as an arrow. (Give examples)
   Consider the system:
$$\frac{dR}{dt} = 1.8R - 1.3RF$$
$$\frac{dF}{dt} = -F + 0.8RF$$

   - Column Vector: $\mathbf{P}(t) = \begin{pmatrix} R(t) \\ F(t) \end{pmatrix}$ (Vectors are boldfaced or have arrows.)
     Then, $\frac{d\mathbf{P}}{dt} = \begin{pmatrix} \frac{dR}{dt} \\ \frac{dF}{dt} \end{pmatrix} = \begin{pmatrix} 1.8R - 1.3RF \\ -F + 0.8RF \end{pmatrix} = \mathbf{V}\begin{pmatrix} R \\ F \end{pmatrix}$
     Examples: For the system above, we have
     - $\mathbf{V}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1.8(0) - 1.3(0)(1) \\ -1 + 0.8(0)(1) \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} =,$
     - $\mathbf{V}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1.8(1) - 1.3(1)(1) \\ -1 + 0.8(1)(1) \end{pmatrix} = \begin{pmatrix} 0.5 \\ -0.2 \end{pmatrix},$
     - $\mathbf{V}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1.8(1) - 1.3(1)(0) \\ 0 + 0.8(1)(0) \end{pmatrix} = \begin{pmatrix} 1.8 \\ 0 \end{pmatrix}.$
   - Row Vector: $\frac{d\mathbf{P}}{dt} = \left(\frac{dR}{dt}, \frac{dF}{dt}\right) = \mathbf{V}(R, F)$
     Example: Consider $\mathbf{V}(2, 1)$ for the example above.
     $\mathbf{V}(2, 1) = (\ 1.8(2) - 1.3(2)(1), -1 + 0.8(2)(1)\ ) = (1, 0.6).$
   - We now can write predator-prey system as $\frac{d\mathbf{P}}{dt} = \mathbf{V}(\mathbf{P})$.

3. We now can consider the right-hand side of a system of 2 differential equations to be a *vector field*, as $\mathbf{V}\begin{pmatrix} R \\ F \end{pmatrix}$ assigns a vector to each point in the RF-plane.
   (Observe that vectors have both a magnitude (length) and a direction-See p. 171, Fig. 2.17.)

4. Sometimes vectors in the vector field overlap, and it becomes sloppy. (See p. 173, Fig. 2.19).
   Alternative: **direction field**-all vectors have same short length. (See p. 173, Fig. 2.20.)
   - Advantage: Easier to look at
   - Disadvantage: Speed is lost-only see direction

5. General Notation: For an arbitrary autonomous system of 2 differential equations

$$\frac{dx}{dt} = f(x, y)$$

$$\frac{dy}{dt} = g(x, y),$$

we can write the system in vector form as follows:

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(\mathbf{Y}),$$

where

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix}, \qquad \text{and} \qquad \mathbf{F}(\mathbf{Y}) = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix}.$$

6. Solutions to a system of differential equations:
If we think of a solution $\mathbf{Y}(t) = (x(t), y(t))$ as a parametrization of a curve in the xy-plane, then $\frac{d\mathbf{Y}}{dt}$ yields the tangent vectors of the curve. Therefore, the equation $\frac{d\mathbf{Y}}{dt} = \mathbf{F}(\mathbf{Y})$ says that the tangent vectors for the solution curves are given by the vectors of the vector field.

7. Example: (You should use HPGSystemsolver to look at the direction field & phase portrait.)

$$\frac{dx}{dt} = x$$

$$\frac{dy}{dt} = 1$$

8. Example: (You should use HPGSystemsolver to look at the direction field & phase portrait.)

$$\frac{du}{dt} = u - 1$$

$$\frac{dv}{dt} = v - 1$$

9. Equilibrium Solutions:
The point $\mathbf{Y}_0$ is an **equilibrium point** for the system $\frac{d\mathbf{Y}}{dt} = \mathbf{F}(\mathbf{Y})$ if $\mathbf{F}(\mathbf{Y}_0) = 0$. The constant function $\mathbf{Y}(t) = \mathbf{Y}_0$ is an **equilibrium solution**.

10. Example: Determine the equilibrium points of the system

$$\frac{dx}{dt} = 2x - y$$
$$\frac{dy}{dt} = x + y.$$

Also look at the phase portrait using HPGSystemSolver.
(In Section 2.3, we'll see that distinct solution curves will never touch or cross.)

Solution: (x_0, y_0) is an equilibrium point of the system above if and only if

$$2x_0 - y_0 = 0, \text{ and}$$
$$x_0 + y_0 = 0.$$

Solving the second equation for x_0, we get $x_0 = -y_0$, and substituting it back into the first, we see

$$2(-y_0) - y_0 = 0 \Rightarrow y_0 = 0.$$

Since $x_0 = -y_0$, it follows that $x_0 = 0$, and the only equilibrium point is $(0, 0)$.

11. Example: Determine the equilibrium points of the system

$$\frac{dR}{dt} = 2R\left(1 - \frac{R}{2}\right) - 1.2RF$$
$$\frac{dF}{dt} = -F + 0.9RF.$$

Also look at the phase portrait in HPGSystemSolver.

Solution:

$$\frac{dR}{dt} = 2R\left(1 - \frac{R}{2}\right) - 1.2RF = R((2 - R) - 1.2F) = R(2 - R - 1.2F)$$
$$\frac{dF}{dt} = -F + 0.9RF = F(-1 + 0.9R).$$

The first equation implies that either $R = 0$ or $2 - R - 1.2F = 0$. If $R = 0$, then the second equation implies that we must have $F = 0$. Hence $(0, 0)$ is the only equilibrium point with $R = 0$.

Conversely, if $F = 0$, then the first equation implies that either $R = 0$ or $2 - R = 0 \Rightarrow R = 2$. It follows that $(2, 0)$ is also an equilibrium point.

Finally, the second equation implies that we could have

$$-1 + 0.9R = 0 \Rightarrow R = 10/9.$$

Substituting this into the first equation, since $R \neq 0$, we have

$$2 - (10/9) - 1.2F = 0,$$

which implies that $F = 20/27$. It follows that $(10/9, 20/27)$ is also an equilibrium point, and these are the only three equilibrium points for this system.