

Section 2.3 - Analytic Methods for Special Systems

1. For systems of differential equations, special forms for which analytic techniques exist are rare. Hence, these special systems are valuable, and we can use them to develop intuition to use when studying systems for which analytic techniques are unavailable.

2. Consider the system

$$\frac{dx}{dt} = 2x + 2y$$

$$\frac{dy}{dt} = x + 3y.$$

For the given functions $\mathbf{Y}(t) = (x(t), y(t))$, check to see if $\mathbf{Y}(t)$ is a solution to the system.

(a) $(x(t), y(t)) = (2e^t, -e^t)$

(b) $(x(t), y(t)) = (3e^{2t} + e^t, -e^t + e^{4t})$

3. A system of differential equations is said to **decouple** if the rate of change of one or more of the dependent variables depends only on its own value. If the equation for $\frac{dx}{dt}$ involves only x and the equation for $\frac{dy}{dt}$ involves only y , we say that the system is **completely decoupled**, and we can solve the two equations separately.

4. Example: A completely decoupled system.

Consider the system

$$\frac{dx}{dt} = 3x$$

$$\frac{dy}{dt} = -2y.$$

Determine the general solution and the particular solution with initial condition $\mathbf{Y}(0) = (1, 4)$.

5. If $\frac{dx}{dt}$ depends on both x and y , but $\frac{dy}{dt}$ depends only on y , we say that the dependent variable y decouples from the system and the system is **partially decoupled**.

6. Example: A partially decoupled system.

Consider the partially decoupled system

$$\frac{dx}{dt} = 3x + 2y$$

$$\frac{dy}{dt} = -y.$$

(a) Derive the general solution to this system.

(b) Determine the solution that satisfies the initial condition $\mathbf{Y}(0) = (1, 0)$.

(c) In the xy -phase plane, plot the solution curve associated to this solution.

(d) Plot the corresponding $x(t)$ - and $y(t)$ -graphs.

7. The Damped Harmonic Oscillator:

The undamped harmonic oscillator equation is

$$m \frac{d^2 y}{dt^2} = -ky,$$

where m is the mass and k is the spring constant. A damping force slows the motion, dissipating energy from the system. The form of the damping force is

$$-b \left(\frac{dy}{dt} \right),$$

where $b > 0$ is called the **coefficient of damping**. To obtain the new model, we equate the product of mass and the acceleration with the sum of the spring force and the damping force, and we get

$$m \frac{d^2 y}{dt^2} = -ky - b \frac{dy}{dt} \text{ or } m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + ky = 0.$$

Equation is called **damped harmonic oscillator**.

Let $p = b/m$ and $q = k/m$ to rewrite the equation as

$$\frac{d^2 y}{dt^2} + p \frac{dy}{dt} + qy = 0.$$

We can convert this into a system by letting $v = \frac{dy}{dt}$ (velocity), and we have

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = -qy - pv.$$

8. Guessing Solutions:

Consider the equation

$$\frac{d^2 y}{dt^2} + 8 \frac{dy}{dt} + 15y = 0.$$

We need $y(t)$ whose second derivative can be expressed in terms of y , $\frac{dy}{dt}$, and constants. We guess $y(t) = e^{st}$. Substitute into LHS of d.e. to get

$$\frac{d^2 y}{dt^2} + 8 \frac{dy}{dt} + 15y = (s^2 + 8s + 15)e^{st}.$$

We get $s = -3$ or $s = -5$ and get solutions $y_1(t) = e^{-3t}$ and $y_2(t) = e^{-5t}$. Determine the general solution.