

## Section 3.1 - Properties of Linear Systems and the Linearity Principle

1. Linear system with constant coefficients:

$$\frac{dx}{dt} = ax + by$$

$$\frac{dy}{dt} = cx + dy,$$

where  $a$ ,  $b$ ,  $c$ , and  $d$  are constants (which may be 0).

(Only first powers of dependent variables)

Other terminology: two-dimensional, linear system with constant coefficients; planar linear systems; linear systems

2. Vector notation: (Matrix times a vector)

- $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  (Coefficient Matrix)

- $\mathbf{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$

- $\mathbf{Y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

- $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix}$

- $\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$

This can be extended to include systems of  $n$  linear equations with  $n$  dependent variables.

3. Equilibrium Points of Linear Systems and the Determinant:

The vector field  $\mathbf{F}(\mathbf{Y}_0)$  at  $\mathbf{Y}_0$  for a linear system is given by

$$\mathbf{F}(\mathbf{Y}_0) = \mathbf{AY}_0$$

Hence, the equilibrium points are the point  $\mathbf{Y}_0$  such that

$$\mathbf{AY}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Clearly,  $(x_0, y_0) = (0, 0)$  is a solution (trivial solution-for **all** linear systems).

- Now assume  $a \neq 0$ . The first equation then implies  $x_0 = -\frac{b}{a}y_0$ .
- The second equation then yields  $c\left(-\frac{b}{a}\right)y_0 + dy_0 = 0$ .

- $(ad - bc)y_0 = 0$
  - Hence,  $y_0 = 0$  or  $ad - bc = 0$ . If  $y_0 = 0$ , then  $x_0 = 0$  and we have trivial solution.
  - Therefore, a linear has a nontrivial solution  $\Leftrightarrow ad - bc = 0$ . (Determinant)
4. Theorem: If  $\mathbf{A}$  is a matrix with  $\det \mathbf{A} \neq 0$ , then the only equilibrium point for the linear system  $\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$  is the origin.
5. Singular/degenerate: ( $\det \mathbf{A} = 0$ ); Nondegenerate: ( $\det \mathbf{A} \neq 0$ )
6. Linearity Principle:  
Suppose  $d\mathbf{Y}/dt = \mathbf{A}\mathbf{Y}$  is a linear system of differential equations.

- (a) If  $\mathbf{Y}(t)$  is a solution of this system and  $k$  is any constant, then  $k\mathbf{Y}(t)$  is also a solution.
- (b) If  $\mathbf{Y}_1(t)$  and  $\mathbf{Y}_2(t)$  are two solutions of this system, then  $\mathbf{Y}_1(t) + \mathbf{Y}_2(t)$  is also a solution.

A solution of the form  $k_1\mathbf{Y}_1(t) + k_2\mathbf{Y}_2(t)$ , where  $k_1$  and  $k_2$  are constants is called a linear combination of the solutions  $\mathbf{Y}_1(t)$  and  $\mathbf{Y}_2(t)$ . Given two solutions, we can produce infinitely many solutions by forming linear combinations of the original two.

$$\frac{d(k\mathbf{Y}_1)}{dt} = k \frac{d\mathbf{Y}_1}{dt} = k\mathbf{A}\mathbf{Y}_1 = \mathbf{A}(k\mathbf{Y}_1)$$

$$\frac{d(\mathbf{Y}_1 + \mathbf{Y}_2)}{dt} = \frac{d\mathbf{Y}_1}{dt} + \frac{d\mathbf{Y}_2}{dt} = \mathbf{A}\mathbf{Y}_1 + \mathbf{A}\mathbf{Y}_2 = \mathbf{A}(\mathbf{Y}_1 + \mathbf{Y}_2)$$

#### 7. Example: Solving Initial-Value Problems

Suppose we want to solve

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix} \mathbf{Y}$$

with initial-value  $\mathbf{Y}(0) = (2, 3)$ . Assume that we are given that

$$\mathbf{Y}_1(t) = \begin{pmatrix} e^{3t} \\ 0 \end{pmatrix} \text{ and } \mathbf{Y}_2(t) = \begin{pmatrix} -e^{-t} \\ 2e^{-t} \end{pmatrix}$$

are solutions to the general equation, but neither is for the initial-value problem. Plug 0 into  $\mathbf{Y}_1$  and  $\mathbf{Y}_2$ , and observe that initial-value problem comes down to solving

$$k_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

#### 8. Linear Independence:

- Expressing arbitrary vectors as linear combinations of vectors is fundamental to linear algebra.
- $(x_1, y_1)$  and  $(x_2, y_2)$  are linearly independent if they do not lie on the same line through the origin or, equivalently, if neither one is a multiple of the other.
- If  $(x_1, y_1)$  and  $(x_2, y_2)$  are linearly independent, then an arbitrary vector can be written as a linear combination of them. (Write out as a Theorem.)

9. Theorem:

Suppose  $\mathbf{Y}_1(t)$  and  $\mathbf{Y}_2(t)$  are solutions of the linear system  $d\mathbf{Y}/dt = \mathbf{A}\mathbf{Y}$ . If  $\mathbf{Y}_1(0)$  and  $\mathbf{Y}_2(0)$  are linearly independent, then for any initial condition  $\mathbf{Y}(0) = (x_0, y_0)$ , we can find constants  $k_1$  and  $k_2$  so that  $k_1\mathbf{Y}_1(t) + k_2\mathbf{Y}_2(t)$  is the solution to the initial-value problem

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}, \quad \mathbf{Y}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

10. Progress: We now know that to find all solutions to a linear system, we need only find two particular solutions with linearly independent initial positions (**Linearly independent solutions**).