Section 3.2 - Straight-Line Solutions

1. Consider the linear system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}, \qquad \text{where } \mathbf{A} = \begin{pmatrix} -1 & 1 \\ 7 & 5 \end{pmatrix}.$$

   First look at the direction field in HPGSystemSolver and observe straight-line solutions.

   - Graphically, we're looking for vectors (x, y) for which the vector field at (x, y) points in the same or opposite direction as the vector from the origin to (x, y).
   - Algebraically, we seek vectors $\mathbf{V} = (x, y)$ such that $\mathbf{AV}$ points in the same or opposite direction as the vector from $(0, 0)$ to (x, y).
   - This is equivalent to finding a nonzero vector $\mathbf{V} = (x, y)$ and number λ such that

$$\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}.$$

     (Such a vector $\mathbf{V}$ is called an eigenvector corresponding to the eigenvalue λ of $\mathbf{A}$.)
     If $\lambda > 0$, vector field points away from origin (same as vector).
     If $\lambda < 0$, vector field points toward origin (opposite vector).
   - Solve for λ in our example above, observing that we get a non-trivial solution if and only if

$$\det\begin{pmatrix} -1-\lambda & 1 \\ 7 & 5-\lambda \end{pmatrix} = 0.$$

2. Eigenvalues and Eigenvectors:
   Given a matrix $\mathbf{A}$, if $\mathbf{V}$ is an eigenvector for the eigenvalue λ, then any scalar multiple $k\mathbf{V}$ is also an eigenvector for λ. (Verify this.)
   Hence, the entire line of vectors through the origin and an eigenvector $\mathbf{V}$ are also eigenvectors for λ.

3. Computing Eigenvalues:

   - Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an arbitrary 2×2 matrix.
   - $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (2 × 2 identity matrix)
   - The characteristic polynomial of a system is the polynomial $\det(A - \lambda I) = 0$. (Polynomial in λ)
   - The roots of the characteristic polynomial of a system are the eigenvalues of the system.
   - In Section 3.4, we look at complex eigenvalues, and in Section 3.5, we look at a single root of multiplicity 2.

4. Computing Eigenvectors:
    - $\mathbf{AV} = \lambda\mathbf{V}$ becomes a system of linear equations in x and y:

      $$\begin{cases} ax + by = \lambda x \\ cx + dy = \lambda y \end{cases}$$

    - Compute the eigenvectors for the aforementioned example.
    - Observe that the equations become redundant for eigenvalues-they're lines.
5. Straight-Line Solutions:
    - Verify that if λ is an eigenvalue with associated eigenvector $\mathbf{V} = (x, y)$, then

      $$\mathbf{Y}(\mathbf{t}) = e^{\lambda t}\mathbf{V} = \begin{pmatrix} e^{\lambda t}x \\ e^{\lambda t}y \end{pmatrix}$$

      is a solution of the differential equation

      $$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}.$$

    - We obtain formulas for straight-line solutions using just the eigenvalues and eigenvectors of the matrix $\mathbf{A}$.
    - If we find two real, distinct eigenvalues λ_1 and λ_2 with eigenvectors $\mathbf{V}_1$ and $\mathbf{V}_2$ respectively, then $\mathbf{V}_1$ and $\mathbf{V}_2$ must be linearly independent.
6. Theorem:
   Suppose the matrix $\mathbf{A}$ has a real eigenvalue λ with associated eigenvector $\mathbf{V}$. Then the linear system $d\mathbf{Y}/dt = \mathbf{AY}$ has the straight-line solution

   $$\mathbf{Y}(t) = e^{\lambda t}\mathbf{V}.$$

   Moreover, if λ_1 and λ_2 are distinct, real eigenvalues with eigenvectors $\mathbf{V}_1$ and $\mathbf{V}_2$, respectively, then the solutions $\mathbf{Y}_1(t) = e^{\lambda_1}\mathbf{V}_1$ and $\mathbf{Y}_2(t) = e^{\lambda_2}\mathbf{V}_2$ are linearly independent and

   $$\mathbf{Y}(t) = k_1 e^{\lambda_1 t}\mathbf{V}_1 + k_2 e^{\lambda_2 t}\mathbf{V}_2$$

   is the general solution of the system.
   (This Theorem allows us to find solutions of linear systems of differential equations using only algebra.)