Section 3.3 - Phase Planes for Linear Systems with Real Eigenvalues

1. Example: Linear systems with both a positive and negative eigenvalue.

$$\frac{dx}{dt} = 2x$$

$$\frac{dy}{dt} = -5y$$

   - Determine eigenvalues, eigenvectors, and general solution.
   - Look at what happens as $t \to \infty$ and as $t \to -\infty$.
   - Look at particular solution with $\mathbf{Y}(0) = (1,1)$. Look at $x(t)$, $y(t)$.
   - Any linear system with one positive eigenvalue and one negative eigenvalue acts as a source in the direction of one eigenvector and as a sink in the direction of the other. An equilibrium point of this form is called a **saddle**.

2. Example: Phase portraits for other saddles.

$$\frac{dx}{dt} = -4x + 4y$$

$$\frac{dy}{dt} = 2x + 3y$$

   - Eigenvalues/Eigenvectors: $\lambda_1 = 4$, $\mathbf{V}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$; $\lambda_2 = -5$, $\mathbf{V}_2 = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$.
   - General Solution: $\mathbf{Y}(t) = k_1 \begin{pmatrix} e^{4t} \\ 2e^{4t} \end{pmatrix} + k_2 \begin{pmatrix} -4e^{-5t} \\ e^{-5t} \end{pmatrix} = \begin{pmatrix} k_1e^{4t} - 4k_2e^{-5t} \\ 2k_1e^{4t} + k_2e^{-5t} \end{pmatrix}$.
   - Plot Straight-Line Solutions: $y = 2x$ $(\lambda = 4, sink)$; $y = -\frac{1}{4}x$ $(\lambda = -5, source)$
   - Draw phase portrait

3. Example: Sinks (Use HPGSystemSolver for remaining examples)

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}, \quad \text{where } \mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}$$

   - General Solution: $\mathbf{Y}(t) = \begin{pmatrix} k_1e^{-t} \\ k_2e^{-3t} \end{pmatrix}$
   - If $k_1 \neq 0$, then solve for e^{-t}, replace in $y(t)$ equation, and we get $y(t) = Kx^3$.
   - All solution curves whose initial conditions are not on the y-axis approach the equilibrium point at the origin along curves that are tangent to the x-axis.

4. More general sinks: (Two distinct negative eigenvalues)

    - Example:
$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}, \quad \text{where } \mathbf{A} = \begin{pmatrix} -2 & -2 \\ 1 & -5 \end{pmatrix}$$
    - Eigenvalues/Eigenvectors: $\lambda_1 = -3$, $\mathbf{V}_1 = (2, 1)$; $\lambda_2 = -4$, $\mathbf{V}_2 = (1, 1)$
    - All solutions with the exception of the straight-line solutions for $\lambda_1 = -4$ tend toward (0,0) tangent to the line of eigenvectors for $\lambda_2 = -3$.
    - For arbitrary sink with eigenvalues $\lambda_1 < \lambda_2 < 0$, all solutions $\rightarrow (0, 0)$ tangent to eigenvectors for λ_2 except those with initial conditions of eigenvectors for λ_2.

5. Sources: (Two distinct positive eigenvalues)

    - Example:
$$\frac{d\mathbf{Y}}{dt} = \mathbf{CY}, \quad \text{where } \mathbf{C} = \begin{pmatrix} 2 & -2 \\ 1 & 5 \end{pmatrix}$$
    - Eigenvalues/Eigenvectors: $\lambda_1 = 3$, $\mathbf{V}_1 = (-2, 1)$; $\lambda_2 = 4$, $\mathbf{V}_2 = (-1, 1)$
    - Look at phase portrait and solution curves. (Away from origin)
    - For arbitrary source with eigenvalues $0 < \lambda_2 < \lambda_1$, all solutions $\rightarrow \infty$ as $t \rightarrow \infty$, and most solution curves leave the origin in the direction of the λ_2-eigenvectors.

6. Stability

    - Sinks are **stable**: nearby initial points yield solutions that tend back toward the equilibrium point as time increases.
    - Saddles and sources are **unstable**: there are initial conditions arbitrarily close to the equilibrium point whose solutions move away.
    - Saddle is a new type of equilibrium point(stable in one direction, unstable in another) –cannot occur in one-dimensional systems.