Section 3.4 - Complex Eigenvalues

1. Complex Numbers
   - $\sqrt{-1} = i$ (Electrical engineers say $\sqrt{-1} = j$, as i stands for current)
   - Complex numbers: $a + bi$, where $a, b \in \mathbb{R}$
   - Any quadratic polynomial (characteristic polynomial) has either two distinct real roots, two quadratic roots(conjugates), or one repeated real root.
   - Euler's Formula:

     $$e^{a+bi} = e^a e^{bi} = e^a(\cos b + i \sin b) = e^a \cos b + ie^a \sin b$$

   - Example:

     $$e^{i\pi} = \cos \pi + i \sin \pi = -1$$

2. Example:

   $$\frac{d\mathbf{Y}}{dt} = \mathbf{AY} = \begin{pmatrix} 2 & 2 \\ -4 & 6 \end{pmatrix} \mathbf{Y}$$

   - Eigenvalues: $4 + 2i$, $4 - 2i$
   - No straight-line solutions
   - Eigenvector for $\lambda_1 = 4 + 2i$: $\mathbf{V}_1 = \begin{pmatrix} 1 \\ 1 + i \end{pmatrix}$ $\quad$ ($x = 1$) in first equation(redundant)
   - It still follows that if λ is a complex eigenvalue for $\mathbf{A}$ with eigenvector $\mathbf{V}$, then $\mathbf{Y}(t) = e^{\lambda t}\mathbf{V}$ is a solution to $\dfrac{d\mathbf{Y}}{dt} = \mathbf{AY}$.
   - $$\begin{aligned} \mathbf{Y}(t) &= \begin{pmatrix} e^{(4+2i)t} \\ (1+i)e^{(4+2i)t} \end{pmatrix} \\ &= \begin{pmatrix} e^{4t}\cos(2t) + ie^{4t}\sin(2t) \\ (1+i)e^{4t}\cos(2t) + i(1+i)e^{4t}\sin(2t) \end{pmatrix} \\ &= \begin{pmatrix} e^{4t}\cos(2t) \\ e^{4t}\cos(2t) - e^{4t}\sin(2t) \end{pmatrix} + i \begin{pmatrix} e^{4t}\sin(2t) \\ e^{4t}\cos(2t) + e^{4t}\sin(2t) \end{pmatrix} \end{aligned}$$

     is a solution.

3. Theorem: Obtaining Real Valued Solutions from Complex Solutions
   Suppose $\mathbf{Y}(t)$ is a complex-valued solution to a linear system

   $$\frac{d\mathbf{Y}}{dt} = \mathbf{AY} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{Y},$$

   with a, b, c, d real. Suppose

   $$\mathbf{Y}(t) = \mathbf{Y}_{\text{re}}(t) + i\mathbf{Y}_{\text{im}}(t),$$

   where $\mathbf{Y}_{\text{re}}$ and $\mathbf{Y}_{\text{im}}(t)$ are real-valued functions of t. Then $\mathbf{Y}_{\text{re}}$ and $\mathbf{Y}_{\text{im}}(t)$ are both solutions of $d\mathbf{Y}/dt = \mathbf{AY}$.

4. For the previous example, we now have two solutions:

   - $\mathbf{Y}_1(t) = \begin{pmatrix} e^{4t}\cos(2t) \\ e^{4t}\cos(2t) - e^{4t}\sin(2t) \end{pmatrix}$
   - $\mathbf{Y}_2(t) = \begin{pmatrix} e^{4t}\sin(2t) \\ e^{4t}\cos(2t) + e^{4t}\sin(2t) \end{pmatrix}$

5. Qualitative Analysis

   - Complex solutions have the form: $\mathbf{Y}(t) = e^{(\alpha+i\beta)t}\mathbf{V}$ $\qquad$ ($\mathbf{V}$ eigenvector)
   - $\mathbf{Y}(t) = e^{\alpha t}(\cos\beta t + i\sin\beta t)\mathbf{V}$
   - $\mathbf{V}$ is constant
   - (a) $e^{\alpha t} \to \infty$ as $t \to \infty$ if $\alpha > 0$ (away from origin-amplitude increases)
     (b) $e^{\alpha t} \to 0$ as $t \to \infty$ if $\alpha < 0$ (toward origin-amplitude decreases)
     (c) $e^{\alpha t} = 1$ if $\alpha = 0$ (amplitude constant)
   - $\sin\beta t$ and $\cos\beta t$ oscillate with period $2\pi/\beta$ (create spiral)
   - Look at phase portrait for example in HPGSystemSolver.

6. Linear systems with complex eigenvalues:
   Give a linear system with eigenvalues $\lambda = \alpha \pm i\beta$, $\beta > 0$, the solutions curves spiral around the origin in the phase plane with **natural period** $2\pi/\beta$.

   - If $\alpha < 0$, then solutions spiral toward origin. (**spiral sink**)
   - If $\alpha > 0$, then solutions spiral away from origin. (**spiral source**)
   - If $\alpha = 0$, then solutions are periodic, same closed curve. (origin=**center**)

   Natural frequency-number of cycles that solutions make in one unit of time.