Section 3.5 - Special Cases: Repeated and Zero Eigenvalues

1. Example: Solve the partially-decoupled system

$$\frac{dx}{dt} = 3x$$

$$\frac{dy}{dt} = 2x + 3y.$$

   - Observe that $\lambda = 3$ is a **repeated eigenvalue** of $\mathbf{A}$ with eigenvector $\mathbf{V} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
   - Solution:

$$\begin{aligned} \mathbf{Y}(t) &= \begin{pmatrix} x_0 e^{3t} \\ 2x_0 t e^{3t} + y_0 e^{3t} \end{pmatrix} \\ &= e^{3t} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + te^{3t} \begin{pmatrix} 0 \\ x_0 \end{pmatrix} \end{aligned}$$

2. Consider a system of the form

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y},$$

where $\mathbf{A}$ has a repeated eigenvalue λ.

   - Assume $\mathbf{Y}(t) = e^{\lambda t}\mathbf{V}_0 + te^{\lambda t}\mathbf{V}_1$ is a solution.
   - Determine $\frac{d\mathbf{Y}}{dt}$ for $\mathbf{Y}(t)$.
   - Calculate the product $\mathbf{A}\mathbf{Y}$.
   - Equate these, observing that the $e^{\lambda t}$ and $te^{\lambda t}$ must be equal.
   - Equating the $te^{\lambda t}$ terms yields

$$\lambda \mathbf{V}_1 = \mathbf{A}\mathbf{V}_1$$

   - Equating the $e^{\lambda t}$ terms yields

$$\lambda \mathbf{V}_0 + \mathbf{V}_1 = \mathbf{A}\mathbf{V}_0, \text{ or}$$

$$\mathbf{V}_1 = (\mathbf{A} - \lambda \mathbf{I})\mathbf{V}_0.$$

3. Theorem:
Suppose $d\mathbf{Y}/dt = \mathbf{A}\mathbf{Y}$ is a linear system in which the 2×2 matrix $\mathbf{A}$ has a repeated real eigenvalue λ but only one line of eigenvectors. Then the general solution has the form

$$\mathbf{Y}(t) = e^{\lambda t}\mathbf{V}_0 + te^{\lambda t}\mathbf{V}_1,$$

where $\mathbf{V}_0 = (x_0, y_0)$ is an arbitrary initial condition and $\mathbf{V}_1$ is determined from $\mathbf{V}_0$ by

$$\mathbf{V}_1 = (\mathbf{A} - \lambda \mathbf{I})\mathbf{V}_0.$$

If $\mathbf{V}_1$ is zero, then $\mathbf{V}_0$ is an eigenvector and $\mathbf{Y}(t)$ is a straight-line solution. Otherwise, $\mathbf{V}_1$ is an eigenvector.

Warning: This doesn't say that $e^{\lambda t}\mathbf{V}_0$ and $te^{\lambda t}\mathbf{V}_1$ are solutions. Remember that $\mathbf{V}_1$ is determined by $\mathbf{V}_0$. Also, $e^{\lambda t}\mathbf{V}_0$ is only a solution if $\mathbf{V}_0$ is an eigenvector.

4. Qualitative Analysis of Systems with Repeated Eigenvalues
   (Look at the graph of the example in HPGSystemSolver)

   - $$\mathbf{Y}(t) = e^{\lambda t}\mathbf{V}_0 + te^{\lambda t}\mathbf{V}_1 = e^{\lambda t}(\mathbf{V}_0 + t\mathbf{V}_1),$$
     where $\mathbf{V}_1 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{V}_0$ is either an eigenvector or zero.
   - If $\lambda < 0$, then as $t \to \infty$, $\mathbf{Y}(t) \to (0,0)$ (sink) along $\mathbf{V}_1$ since $t\mathbf{V}_1$ dominates.
   - If $\lambda > 0$, then $\mathbf{Y}(t) \to \infty$ as $t \to \infty$ along $\mathbf{V}_1$.
   - Solutions try to spiral around the origin but cannot since the straight-line solutions get in the way.
   - Think of repeated eigenvalue case as a bifurcation between 2 distinct real eigenvalue case (2 straight-line solutions) and complex conjugate eigenvalue case (no straight-line solutions.)

5. Example:
   $$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mathbf{Y}$$
   System has repeated eigenvalue $\lambda = a$, and every nonzero vector is an eigenvector for $\lambda = a$.

6. Systems with Zero as an Eigenvalue: (Look at in HPGSystemSolver)
   Consider the system
   $$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}, \qquad \text{where } \mathbf{A} = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}.$$

   - Eigenvalues/Eigenvectors: $\lambda_1 = 0$, $\mathbf{V}_1 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$; $\lambda_2 = 8$, $\mathbf{V}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$
   - General Solution: $\mathbf{Y}(t) = k_1\mathbf{V}_1 + k_2 e^{8t}\mathbf{V}_2$
   - Every point on the line of eigenvectors for $\lambda_1 = 0$ is an equilibrium point ($k_2 = 0$).
   - If $\lambda_2 < 0$, then solution tends to equilibrium point $k_1\mathbf{V}_1$ along a line parallel to $\mathbf{V}_2$.
   - If $\lambda_2 > 0$, then solution moves away from the line of equilibrium points as t increases.