

Section 3.6 - Second-Order Linear Equations

1. Second-Order Equations versus First-Order Systems: Harmonic Oscillator ($m > 0, k > 0$)

$$m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + ky = 0, \text{ or}$$

$$\frac{d^2 y}{dt^2} + p \frac{dy}{dt} + qy = 0$$

Convert this to a system of first-order linear system.

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \mathbf{Y}$$

In general, the following is a **linear, homogeneous** second-order equation.

$$a \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + cy = 0$$

2. Example: Shortcut method for finding the general solution of a linear second-order equation.

$$\frac{d^2 y}{dt^2} + 7 \frac{dy}{dt} + 12y = 0$$

We know that solutions to this are of the form $\mathbf{Y}(t) = e^{\lambda t} \mathbf{V}$ (eigenvalue/eigenvector). Guess $y(t) = e^{\lambda t}$.

$$\frac{d^2 y}{dt^2} + 7 \frac{dy}{dt} + 12y = \lambda^2 e^{\lambda t} + 7\lambda e^{\lambda t} + 12e^{\lambda t}.$$

Since $e^{\lambda t}$ is never zero, we must have

$$\lambda^2 + 7\lambda + 12 = 0.$$

This is the characteristic equation. The Linearity Principle says that the general solution is

$$y(t) = k_1 e^{-3t} + k_2 e^{-4t}.$$

Relate back to linear system and general solution there.

3. Example: Solve the initial-value problem

$$\frac{d^2 y}{dt^2} + 7 \frac{dy}{dt} + 12y = 0, \quad y(0) = 5, \quad y'(0) = -11.$$

4. Example: Complex Eigenvalues

$$\frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} + 5y = 0.$$

Solve as both a second-order linear equation and a first-order linear system.

5. The Undamped Harmonic Oscillator

$$m \frac{d^2 y}{dt^2} + ky = 0.$$

Complex solution is

$$e^{i\omega t} = \cos \omega t + i \sin \omega t.$$

General solution is

$$y(t) = k_1 \cos \omega t + k_2 \sin \omega t, \text{ or}$$
$$\mathbf{Y}(t) = k_1 \begin{pmatrix} \cos \omega t \\ -\omega \sin \omega t \end{pmatrix} + k_2 \begin{pmatrix} \sin \omega t \\ \omega \cos \omega t \end{pmatrix}.$$

The period is $2\pi/\omega$, so the **natural frequency** is $\omega/2\pi$.

6. Harmonic Oscillators with Damping:

$$m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + ky = 0$$

Characteristic Equation is

$$m\lambda^2 + b\lambda + k = 0$$

with roots

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}.$$

Three possibilities for roots of the characteristic equation.

- If $b^2 - 4km < 0$, then we have complex roots. The real part of these roots is $-b/2m$, which is always negative. The harmonic oscillator is said to be **underdamped**.
- If $b^2 - 4km > 0$, then there are two distinct real roots to this equation. The oscillator is said to be **overdamped**.
- If $b^2 - 4km = 0$, we have repeated roots and the oscillator is **critically damped**.

7. Example: Harmonic oscillator with $m = 1$, $k = 8$, $b = 6$, $y(0) = 1$, $v(0) = 0$.

- (a) Look at second-order differential equation and first-order system.
- (b) Solve both.
- (c) Classify as underdamped, overdamped, critically damped, undamped.
- (d) Sketch the phase portrait, $y(t)$, and $v(t)$ graphs.