Section 4.1 - Forced Harmonic Oscillators

1.
   - Homogeneous - Right-hand side is zero
   - Nonhomogeneous - Right-hand side is nonzero
   - Forced harmonic oscillator (Forced equation-$f(t)$ is new, external force) $m > 0$, $k > 0$, $b \geq 0$

   $$m\frac{d^2y}{dt^2} = -ky - b\frac{dy}{dt} + f(t)$$

   $$\frac{d^2y}{dt^2} + \frac{b}{m}\frac{dy}{dt} + \frac{k}{m}y = \frac{f(t)}{m}$$

   $$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = g(t).$$

   Second-order, linear, nonhomogeneous, nonautonomous.

   - $$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0$$

   is the corresponding homogeneous unforced equation.

2. Guess and Check method for solving $\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0.$

   Example: $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 0.$

   Guess: $y(t) = e^{st} \Rightarrow s^2 + 6s + 8 = 0.$

   This is just the characteristic polynomial, and hence the general solution is $y(t) = k_1e^{-2t} + k_2e^{-4t}$.

3. Extended Linearity Principle:

   Consider a nonhomogeneous equation

   $$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = g(t)$$

   and its corresponding homogeneous equation

   $$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0.$$

   - If $y_p(t)$ is a particular solution of the nonhomogeneous equation and $y_h(t)$ is a solution of the homogeneous equation, then $y_h(t) + y_p(t)$ is also a solution to the nonhomogeneous equation. (i.e. if $k_1y_1(t) + k_2y_2(t)$ is the general solution to the homogeneous equation, then $k_1y_1(t) + k_2y_2(t) + y_p(t)$ is a solution to the nonhomogeneous equation.)
   - If $y_p(t)$ and $y_q(t)$ are two solutions of the nonhomogeneous equation, then $y_p(t) - y_q(t)$ is a solution of the corresponding homogeneous equation. (i.e. Any solution to the nonhomogeneous equation can be written as $k_1y_1(t)+k_2y_2(t)+y_p(t)$.)

   Hence, if $k_1y_1(t) + k_2y_2(t)$ is the general solution of the homogeneous equation, then

   $$k_1y_1(t) + k_2y_2(t) + y_p(t)$$

   is the general solution of the nonhomogeneous equation. (Verify the 2 bullet points.)

4. Finding the general solution for forced harmonic oscillators:

   (1) Find the general solution of the homogeneous second-order equation.

   (2) Find *one* particular solution of the nonhomogeneous second-order equation.

   (3) Add the results of the previous two steps to obtain the general solution of the forced equation.

   Observe that if the damping coefficient p is positive(as in any physical, mechanical device), then the equilibrium point at the origin is a sink, and consequently all solutions approach $y_p(t)$ for large t.

$$\frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

   $p^2 - 4q < 0 \Rightarrow$ spiral sink
   $p^2 - 4q = 0 \Rightarrow$ repeated negative eigenvalue (sink)
   $p^2 - 4q > 0 \Rightarrow$ 2 distinct negative eigenvalues (sink)

5. Steady-state solution

   - $y_p(t)$ is often called the **forced response**. ($y_p(t)$ need not be constant)
   - The **steady-state response** describes the behavior of the forced response for large t.
   - A solution to the unforced (homogeneous) harmonic oscillator is called the **natural response**.

6. Example: Method of Undetermined Coefficients:
   Find the general solution of the differential equation

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 2e^{-3t}.$$

   ($y_h(t) = k_1 e^{-2t} + k_2 e^{-4t}$; $\quad y_p(t) = -2e^{-3t}$)

7. Example:

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = e^{-2t}$$

   - Observe that our initial guess $y_p(t) = e^{-t}$ does not work.
   - As with repeated eigenvalue case, our next guess is $y(t) = kte^{-2t}$.
   - We now determine k. ($k = 1/2$)