Section 4.2 - Sinusoidal Forcing

1. We will study the forced harmonic oscillator equation
$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = g(t),$$
where $g(t)$ is a sine or cosine function.
Examples: earthquake shaking a building, pressure waves of sound striking glass

2. Dividing Complex Numbers:
$$\frac{1}{a+bi} = \frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2}$$
Example: Solve $(1+3i)a = 1$.

3. Consider
$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = e^{it}.$$
If $y_c(t)$ is a particular solution to this complex differential equation, then
$$y_c(t) = y_{\text{re}}(t) + iy_{\text{im}}(t).$$
Substituting $y_c(t)$ into the differential equation, using Euler's formula, and equating real and imaginary parts of both sides, we get
$$\frac{d^2y_{\text{re}}}{dt^2} + 3\frac{dy_{\text{re}}}{dt} + 2y = \cos t \qquad \text{and} \qquad \frac{d^2y_{\text{im}}}{dt^2} + 3\frac{dy_{\text{im}}}{dt} + 2y = \sin t$$
Hence, the real part is a particular solution of the original equation with forcing $\cos t$, and the imaginary part is a particular solution of the original equation with forcing $\sin t$.

4. Example:
$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \sin t$$
The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -2$, and hence $y_h(t) = k_1e^{-t} + k_2e^{-2t}$ is the general solution to the homogeneous equation. We now only need a particular solution to the non-homogeneous equation.
Now, similar to Section 4.1, we guess $y_c(t) = ae^{it}$. We get $a = \dfrac{1-3i}{10}$. Hence,
$$y_c(t) = \frac{1-3i}{10}(\cos t + i\sin t) = \left(\frac{1}{10}\cos t + \frac{3}{10}\sin t\right) + i\left(-\frac{3}{10}\cos t + \frac{1}{10}\sin t\right).$$
Since the right-hand side our equation is $\sin t$, we take the imaginary part and obtain
$$y_p(t) = -\frac{3}{10}\cos t + \frac{1}{10}\sin t.$$
The general solution of the differential equation consequently is
$$y(t) = k_1e^{-t} + k_2e^{-2t} + -\frac{3}{10}\cos t + \frac{1}{10}\sin t.$$

5. Magnitude of Complex Numbers:

$$|a+bi| = \sqrt{a^2+b^2}$$

Example:

$$\left|\frac{1}{1+3i}\right| = \left|\frac{1-3i}{10}\right| = \frac{\sqrt{10}}{10}$$

6. Qualitative Analysis (Look at $y(t)$ graphs in HPGSystemSolver)
In HPGSystemSolver, look at

$$\frac{dy}{dt} = v,$$

$$\frac{dv}{dt} = -3v - 2y + \sin t.$$

The portion of a solution arising from the solution to the unforced oscillator (homogeneous) tends to zero like e^{-t}. Hence, for large t, every solution is close to the particular solution

$$y_p(t) = -\frac{3}{10}\cos t + \frac{1}{10}\sin t.$$

All solutions have the same long-term behavior including the long-term amplitude of the oscillations. For our example, we have

$$|y(t)| = |a| = \frac{\sqrt{10}}{10}.$$