

Section 4.3 - Undamped Forcing and Resonance

1. Undamped Harmonic Oscillator:

$$m \frac{d^2 y}{dt^2} + ky = 0$$

We'll deal only with $m = 1$.

2. Complex Eigenvalues

- Consider the second-order differential equation $\frac{d^2 y}{dt^2} + 4 \frac{dy}{dt} + 13y = 0$.
- Eigenvalues are $-2 \pm 3i$.
- Complex solution is $y(t) = e^{(-2+3i)t} = e^{-2t}(\cos 3t + i \sin 3t)$
- We have two real solutions: $y_1(t) = e^{-2t} \cos 3t$ and $y_2(t) = e^{-2t} \sin 3t$.
- The general solution is: $y(t) = k_1 e^{-2t} \cos 3t + k_2 e^{-2t} \sin 3t$.

3. Consider a particular harmonic oscillator equation with sinusoidal forcing given by

$$\frac{d^2 y}{dt^2} + 2y = \cos \omega t.$$

The parameter ω controls the frequency of the external forcing. (Frequency = $\frac{1}{\text{period}}$.)

Forcing frequency = $\omega/2\pi$.

- General solution to the unforced equation is $y_h(t) = k_1 \cos \sqrt{2}t + k_2 \sin \sqrt{2}t$.
Natural frequency = $\sqrt{2}/(2\pi)$.
- Use the Method of Undetermined Coefficients to find one solution of the forced equation.

$$y_c(t) = \frac{1}{2 - \omega^2} e^{i\omega t} = \frac{1}{2 - \omega^2} (\cos \omega t + i \sin \omega t)$$

$$y_p(t) = \frac{1}{2 - \omega^2} \cos \omega t$$

- General Solution:

$$y(t) = k_1 \cos \sqrt{2}t + k_2 \sin \sqrt{2}t + \frac{1}{2 - \omega^2} \cos \omega t$$

Observe that $y(t)$ is undefined at $\omega = \sqrt{2} \approx 1.4$.

4. Qualitative Analysis: (Look at in HPGSystemSolver as ω varies from 0.5 to 1.2)

- **Beating:** occurs when the natural response and forced response have approximately the same frequency.
- Frequency of the beats = $\frac{\text{natural frequency} - \text{forced frequency}}{2}$. (Discussion in text.)
- Frequency of rapid oscillations = $\frac{\text{natural frequency} + \text{forced frequency}}{2}$.
- Determine these frequencies for our example with $\omega = 1.1$. (Look at in HPGSystemSolver.)
Period of beats ≈ 40 ; Period of rapid oscillation ≈ 5 .

5. Resonance:

- Forcing whose frequency is the same as the natural frequency of the oscillator is called **resonant forcing**, and the oscillator is said to be in **resonance**.
- Look at what happens in the general solution of our example as $\omega \rightarrow \sqrt{2}$.
- Compute the general solution in the resonant case:

$$\frac{d^2y}{dt^2} + 2y = \cos \sqrt{2}t.$$

General solution: $y(t) = k_1 \cos \sqrt{2}t + k_2 \sin \sqrt{2}t + \frac{1}{2\sqrt{2}} t \sin \sqrt{2}t.$

- Point out Millennium Bridge example and Tacoma Narrows Bridge examples.