Section 6.1 - Laplace Transforms

1. Example: Review of Partial Fraction Decomposition (Calc II):
   Rewrite $\dfrac{4}{s^2-1} = \dfrac{A}{s-1} + \dfrac{B}{s+1}$.

2. For this section, we assume that:
   (a) $y(t)$ is continuous or piecewise-continuous
   (b) We can find real numbers M, k for which
   $$|y(t)| \leq e^{Mt},$$
   if $t \geq k$. In other words, $y(t)$ has no more than exponential growth.

3. What is a Linear Operator? We say that σ is a linear operator if:
   (a) $\sigma(\alpha + \beta) = \sigma(\alpha) + \sigma(\beta)$
   (b) $\sigma(k\alpha) = k\sigma(\alpha)$ for all constants k

4. Examples of Linear Operators:
   - Derivative
   - Indefinite Integral (Antiderivative)

5. Let $y(t)$ be a continuous function. We may want to compare $y(t)$ to a well-known function like a trig function, a polynomial function, or an exponential function.

6. Laplace Tranform: Compare $y(t)$ to $f(t) = e^{-st}$
   - Define $Y(s) = \displaystyle\int_0^\infty y(t)e^{-st}\, dt$. (Laplace Transform of $y(t)$.)
   - We sometimes write $Y(s) = \mathcal{L}[y(t)]$ or $Y = \mathcal{L}[y]$.

7. Example: Find the Laplace Transform of e^{3t}.
$$\begin{aligned} Y(s) &= \int_0^\infty e^{3t}e^{-st}\, dt \\ &= \int_0^\infty e^{t(3-s)}\, dt \\ &= \lim_{b\to\infty} \int_0^b e^{t(3-s)}\, dt \\ &= \lim_{b\to\infty} \frac{1}{3-s} e^{t(3-s)}\Big|_0^b \\ &= \frac{1}{3-s} \lim_{b\to\infty} ((e^{3-s})^b - 1). \end{aligned}$$
   - If $s > 3$, we get $Y(s) = \dfrac{1}{3-s}(0-1) = \dfrac{1}{s-3}$. (Converges)
   - If $s \leq 3$, $Y(s)$ is not defined.

- We say that $\mathcal{L}[e^{3t}] = \dfrac{1}{3-s}$ if $s > 3$.
- We can generalize this for any constant a to get $\mathcal{L}[e^{at}] = \dfrac{1}{s-a}$ if $s > a$.

8. Example: Determine $\mathcal{L}[c]$

9. Properties of the Laplace Transform:

   - $\mathcal{L}[f(t)+g(t)] = \int_0^\infty [f(t)+g(t)]e^{-st}\,dt = \int_0^\infty f(t)e^{-st}\,dt + \int_0^\infty g(t)e^{-st}\,dt$
     $= \mathcal{L}[f(t)] + \mathcal{L}[g(t)]$.
   - Similarly, $\mathcal{L}[kf(t)] = k \cdot \mathcal{L}[f(t)]$.
   - Thus, $\mathcal{L}$ is a **linear operator.**

10. Show that $\mathcal{L}\left[\dfrac{dy}{dt}\right] = s \cdot \mathcal{L}[y(t)] - y(0)$.

$$\begin{aligned}
\mathcal{L}\left[\frac{dy}{dt}\right] &= \int_0^\infty \frac{dy}{dy}\, e^{-st}\,dt \\
&= \lim_{b\to\infty} \left(e^{-st}y(t)\Big|_0^b - \int_0^\infty -se^{-st}y(t)\,dt\right) && \text{(Integration by Parts)} \\
&= \lim_{b\to\infty} \left(e^{-sb}y(b) - e^0 y(0)\right) + s\int_0^\infty e^{-st}y(t)\,dt
\end{aligned}$$

Because $|y(b)| \le e^{Mb}$ as $b \to \infty$ (assumed), we know $\lim_{b\to\infty} e^{-sb}y(b) = 0$.

So, $\mathcal{L}\left[\dfrac{dy}{dt}\right] = 0 - y(0) + s\int_0^\infty e^{-st}y(t)\,dt$, or

$$\mathcal{L}\left[\frac{dy}{dt}\right] = s \cdot \mathcal{L}[y(t)] - y(0).$$

11. Inverse Laplace Transform:

$$\mathcal{L}^{-1}[F] = f \qquad \text{if and only if} \qquad \mathcal{L}[f] = F.$$

Since the Laplace Transform is a linear operator, it following the the Inverse Laplace transform is also.

   - $\mathcal{L}^{-1}[f+g] = \mathcal{L}^{-1}[f] + \mathcal{L}^{-1}[g]$
   - $\mathcal{L}^{-1}[cf] = c \cdot \mathcal{L}^{-1}[f]$
   - $\mathcal{L}^{-1}\left[\dfrac{1}{s-a}\right] = e^{at}$ for $s > a$

12. Example: Find $\mathcal{L}^{-1}\left[\dfrac{5}{(s-1)(s-2)}\right]$

13. Example: Consider the function $\dfrac{dy}{dt} + 9y = 2$, $y(0) = -2$.

    (a) Compute the Laplace transform of both sides of the equation.

(b) Substitute the initial conditions and solve for the Laplace transform of the solution.

(c) Find a function whose Laplace transform is the same as the solution (Inverse Laplace)

(d) Check your solution.