

## Appendix B - Taylor Series

1. A First-order Example:

$$\frac{dy}{dt} = t^2y + 1$$

Linear, so we could try an integrating factor. (Integrals problematic)

- Guess: Any good guesses?
- Guess Taylor series for  $y(t)$  centered at  $t = 0$  (Maclaurin series)

$$y(t) = a_0 + a_1t + a_2t^2 + \dots = \sum_{n=0}^{\infty} a_n t^n.$$

- We ignore questions of convergence: formal analysis.

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$$\frac{dy}{dt} = a_1 + 2a_2t + 3a_3t^2 + \dots = \sum_{n=1}^{\infty} n a_n t^{n-1}$$

- Differential equation becomes

$$\begin{aligned} a_1 + 2a_2t + 3a_3t^2 + 4a_4t^3 + 5a_5t^4 + \dots &= t^2(a_0 + a_1t + a_2t^2 + a_3t^3 + \dots) + 1 \\ &= 1 + a_0t^2 + a_1t^3 + a_2t^4 + a_3t^5 + \dots \end{aligned}$$

- The 2 series are equal  $\Leftrightarrow$  corresponding coefficients are equal. (Infinite list of equations)
- $a_1 = 1, \quad 2a_2 = 0, \quad 3a_3 = a_0, \quad 4a_4 = a_1, \quad 5a_5 = a_2, \quad \dots$
- Rework this into polynomial approximation solutions, then a power series solution.

2. A Second-order Example: Hermite's equation (arises in physics)

$$\frac{d^2y}{dt^2} - 2t \frac{dy}{dt} + 2py = 0,$$

where  $p$  is a parameter.

- Guess  $y(t) = a_0 + a_1t + a_2t^2 + \dots = \sum_{n=0}^{\infty} a_n t^n.$

- This implies that  $\frac{dy}{dt} = a_1 + 2a_2t + 3a_3t^2 + 4a_4t^3 + \dots = \sum_{n=1}^{\infty} n a_n t^{n-1}.$

- Hence  $\frac{d^2y}{dt^2} = 2a_2 + 6a_3t + 12a_4t^2 + \dots = \sum_{n=1}^{\infty} n(n-1)a_n t^{n-2}.$

- Note that  $y(0) = a_0$  and  $y'(0) = a_1.$

- Substituting these series into Hermite's equation, we get

$$(2a_2 + 6a_3t + 12a_4t^2 + \dots) - 2t(a_1 + 2a_2t + 3a_3t^2 + 4a_4t^3 + \dots) + 2p(a_0 + a_1t + a_2t^2 + \dots) = 0.$$

- $(2pa_0 + 2a_2) + (2pa_1 - 2a_1 + 6a_3)t + (2pa_2 - 4a_2 + 12a_4)t^2 + (2pa_3 - 6a_3 + 20a_5)t^3 + \dots = 0.$

- All coefficients must be zero, so we have

$$2pa_0 + 2a_2 = 0, \quad 2(p-1)a_1 + 6a_3 = 0, \quad 2(p-2)a_2 + 12a_4 = 0, \quad 2(p-3)a_3 + 20a_5 = 0, \dots$$

- Since  $a_0 = y(0)$  and  $a_1 = y'(0)$  are determined by initial conditions, we can get other coefficients in terms of  $a_0$  and  $a_1$ .
- Look at when  $p = 0$ ,  $p = 1$  ( $y(0), y'(0) = (1, 0)$ ), and when  $p = 2$  ( $y(0), y'(0) = (0, 1)$ ).
- Similar results hold for IVP with same initial conditions for odd/even  $p$ . These solutions are called the **Hermite polynomials**  $H_p(t)$ .