Chapter 2: Definitions and Examples

2 Definitions

1. Definitions:

   - **Vertex Set** $(V(G))$
   - **Edge Family** $(E(G))$
   - Edge $\{v, w\}$ or vw **joins** v to w
   - **Simple graph:** Non-empty finite set $V(G)$ and finite set $E(G)$ of distinct unordered pairs of distinct elements of $V(G)$
   - **Graph:** allows loops and multiple edges; $E(G)$ (**edge family**) is a finite family of unordered pairs of (not necessarily distinct) elements of $V(G)$
   - In this book, all graphs are finite and undirected, with loops and multiple edges allowed unless specifically excluded.
   - Two graphs G_1 and G_2 are **isomorphic** if we can pair up each vertex in $V(G_1)$ with a vertex in $V(G_2)$ in such a way that any two vertices in $V(G_1)$ are connected by the same number of edges as their corresponding vertices in $V(G_2)$.

     If G_1 and G_2 are simple and $v_1, v_2 \in V(G_1)$ are paired up with $w_1, w_2 \in V(G_2)$, respectively, then $v_1v_2 \in E(G_1) \Leftrightarrow w_1w_2 \in E(G_2)$.

     Example: p. 15, pr. 2.3.
   - Graphs (vertices) may be 'labelled' or 'unlabelled'
   - **Connected**, **Disconnected**, **Component**, **Union**
   - p.11 lists all connected unlabelled graphs with up to five vertices

2. Example: p. 15, pr. 2.5.

3. Other Definitions:

   - **Adjacent vertices** and a vertex **incident** to an edge
   - **Isolated vertex** (degree 0) and **end-vertex** (degree 1)
   - **Degree sequence** - degrees written in increasing order

4. **Handshaking Lemma** (Euler 1735): If several people shake hands, then the total number of hands shaken must be even, as two hands are involved in each handshake.
   Corollary: In any graph, the number of vertices of odd degree is even.

5. More Definitions:

   - **Subgraph**
   - Deletion: $G - v$
   - Contraction: $G \backslash e$

6. Matrix representations: G is a graph with vertices $\{1, 2, \ldots, n\}$.

   - **Adjacency matrix**: $n \times n$ matrix whose $ij-$th entry is the number of edges joining vertex i to vertex j.
   - **Incidence matrix**: $n \times m$ matrix whose $ij-$th entry is 1 if vertex i is incident to edge j, and 0 otherwise.

3 Examples

1. Examples:

   - **Null graph** (N_n): n vertices, no edges
   - **Complete graph** (K_n): n vertices, $n(n-1)/2$ edges

   - **Regular graph**: every vertex has the same degree; **regular of degree** r or r**-regular**

   - **Cycle graph**: (C_n): Connected, regular graph with n vertices

   - **Path graph**: (P_n): n vertices, obtained by removing one edge from C_n

   - **Wheel**: (W_n): n vertices, joining a vertex v to each vertex of C_{n-1}

   - **Cubic graphs**: regular of degree 3;
     Example: **Petersen graph**: Star inside pentagon

   - **Platonic graphs** (regular): tetrahedron, octahedron, cube, icosahedron, dodecahedron
   - **Bipartite graph**: $V(G)$ can be split into two disjoint sets A and B so that any edge in $E(G)$ goes from a vertex in A to a vertex in B.

   - **Complete bipartite graph**: $K_{m,n}$, $m+n$ vertices, mn edges

   - k**-cube** (Q_k): vertices correspond to sequence $(a_1, a_2, \ldots, a_k)$, where each a_i equals 0 or 1, and whose edges join those sequences that differ in one place; 2^k vertices and $k2^{k-1}$ edges, k-regular

   - **Complement** ($\bar{G}$) of a simple graph

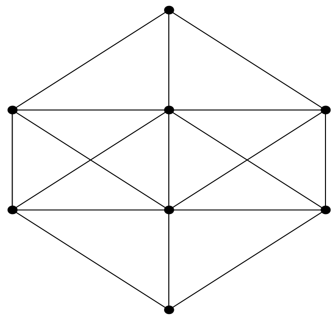
2. Examples to work out:

   (a) p. 20, pr. 3.3

   (b) p. 20, pr. 3.4

4 Three Puzzles

1. The Eight Circles Problem ($8! = 40,320$ possibilities)



   - Place the letters A, B, C, D, E, F, G, H into the eight circles in such a way that no letter is adjacent to a letter that is next to it in the alphabet.
   - Easiest letters are A and H, since there only next to one letter.
   - Hardest circles are those in the middle, since there adjacent to six others.

2. Six People at a Party
   Show that, in any gathering of six people, there are either three people who all know each other or three people none of whom knows either of the other two.
   - Draw a graph with six vertices.
   - Vertices are connected by a solid edge if the two people know each other.
   - Vertices are connected by a dotted edge if the two people don't know each other.
   - Any vertex v has degree 5. At least three of these edges must be of the same type. Assume dotted. Call the edges at the other end w, x, and y.
   - If w and x, w and y, or x and y don't know each other, respectively, then that edge would form the third edge of a dotted triangle. If, however, x, y, and z all know each other then they form a solid triangle.

3. The Four Cubes Problem
   Given four cubes whose faces are coloured red, blue, green, and yellow, can we pile them up so that all four colours appear on each side of the resulting 4×1 *stack?*
   - Represent each cube by a graph with 4 vertices R, B, G, and Y.
   - Two vertices are adjacent if and only if the cube has the corresponding colours on opposite faces.
   - Superimpose the graphs to form a new graph G. (Number edges for cube)
   - We need to find subgraphs H_1 (front and back) and H_2 (left and right) such that:
     (a) Each subgraph contains exactly one edge from each cube: they tell us which colours appear front & back, left & right.
     (b) The subgraphs have no edges in common: ensures that the faces on front & back are different than those on left & right.
     (c) Each subgraph is 2-regular: ensures that each colour appears exactly once each on front, back, left, and right.