

Chapter 3: Paths and Cycles

5 Connectivity

1. Definitions:

- **Walk:** finite sequence of edges in which any two consecutive edges are adjacent or identical. (**Initial vertex, Final vertex, length**)
- **Trail:** walk with all distinct edges
- **Path:** trail with distinct vertices
- **Closed:** first and last vertices the same
- **Cycle:** closed path with at least one edge **Triangle**
- **Connected:** there is a path between each pair of vertices

2. Example: p. 30, pr. 5.4

Prove that a simple graph and its complement cannot both be disconnected.

3. **Theorem:** G is a bipartite graph \Leftrightarrow each cycle of G has even length.

pf. \Rightarrow Since G is bipartite, we can split $V(G)$ into disjoint sets A and B so that each edge of G joins a vertex of A and a vertex of B .

Let $v_0 \rightarrow v_1 \rightarrow \dots \rightarrow v_m \rightarrow v_0$ be a cycle in G . (Assume WLOG that $v_0 \in A$.)

Then $v_1 \in B, v_2 \in A, \dots, v_m \in B \Rightarrow$ cycle length is even.

\Leftarrow Assume that every cycle of G has even length.

First note that if G is disconnected, then it is sufficient to show that every component is bipartite. We may hence assume that G is connected.

Pick $u \in G$. Let A be the set of all vertices v whose shortest uv -path has even length, and let B be the set of all vertices v whose shortest uv -path has odd length. Clearly $u \in A$; $V(G) = A \cup B$; and $A \cap B = \emptyset$.

We need only show that any edge in $E(G)$ must connect a vertex of A to one of B .

To obtain a contradiction, we assume that v and w are adjacent vertices of A .

Let $P = \{u \rightarrow v_1 \rightarrow \dots \rightarrow v_{2p} = v\}$ and $Q = \{u \rightarrow w_1 \rightarrow \dots \rightarrow w_{2q} = w\}$ be shortest paths from u to v and w , respectively. Let u' be the last vertex that is contained in both paths. (u' may equal u and hence exists.)

Then, the parts of P and Q from u to u' are both shortest paths.

Since these paths have the same length, i , we have $u' = v_i = w_i$.

It follows that $C = \{u_i \rightarrow u_{i+1} \rightarrow \dots \rightarrow v \rightarrow w \rightarrow w_{2q-1} \rightarrow \dots \rightarrow w_i\}$ is an odd cycle. (Consider separately cases of i odd and even.)

This is a contradiction. The result follows similarly if we assume that there exists an edge connecting vertices of B . Hence G is bipartite.

4. **Theorem:** Let G be a simple graph on n vertices. If G has k components, then the number, m , of edges of G satisfies

$$n - k \leq m \leq (n - k)(n - k + 1)/2.$$

pf. We'll prove the upper bound. (Book proves both upper and lower bounds.)

- Induction on the number of edges of G (trivial if G is null graph).
 - Assume that if a graph has $m < m_0$ edges, then $m \geq n - k$.
 - Let G have m_0 edges, k_0 components, and n_0 vertices.
 - If removing an edge doesn't create a new component, then the resulting graph has $m_0 - 1$ edges, k_0 components, and n_0 vertices.
 - By induction hypothesis, $m_0 - 1 \geq n_0 - k_0$, which implies that $m_0 \geq n_0 - k_0 + 1 \geq n_0 - k_0$.
 - If removing an edge of G increases the number of components by 1, then the remaining graph has n_0 vertices, $k_0 + 1$ components, and $m_0 - 1$ edges.
 - By our induction hypothesis $m_0 - 1 \geq n_0 - (k_0 + 1)$, which implies that $m_0 \geq n_0 - k_0$.
5. **Corollary:** Any simple graph with n vertices and more than $\frac{(n-1)(n-2)}{2}$ edges is connected.
6. Other definitions:

- **Disconnecting set** (of a connected graph G): set of edges that disconnects G
- **cutset:** a disconnecting set (of edges) for which no proper subset is a disconnecting set
- Also define for disconnected graphs
- **bridge:** a cutset consisting of only one edge
- **edge connectivity** ($\lambda(G)$): the size of the smallest cutset in G
- **k -edge connected:** G is k -edge connected if $\lambda(G) \geq k$.
- We need analogous concepts for removal of vertices, first for connected graphs.
- **Separating set:** a set of vertices whose deletion disconnects G
- **cut-vertex:** a separating set that contains only one vertex v
- **connectivity** ($\kappa(G)$): the size of the smallest separating set in G .
- **k -connected:** G is k -connected if $\kappa(G) \geq k$.
- Note: It can be proved that if G is connected, then $\kappa(G) \leq \lambda(G)$.

7. Example: p. 30, pr. 5.11

- (a) Prove that, if two distinct cycles of a graph G each contain an edge e , then G has a cycle that does not contain e .
- (b) Prove a similar result with 'cycle' replaced by 'cutset'.

8. Example: p. 30, pr. 5.13

A set E of edges of a graph G is **independent** if E contains no cycle of G . Prove that

- (a) any subset of an independent set is independent.
- (b) if I and J are independent sets of edges with $|J| > |I|$, then there is an edge e that lies in J but not in I with the property that $I \cup \{e\}$ is independent.

Show also that (i) and (ii) still hold if we replace the word 'cycle' with 'cutset'.

6 Eulerian Graphs

1. Definitions:

- (a) **Königsberg Bridge Problem** (Euler 1736) - cross each of 7 bridges exactly once and return to your starting point
- (b) An **Eulerian trail** is a closed trail containing every edge of G . A graph G is **Eulerian** if it contains an Eulerian trail and **semi-Eulerian** if there exists a trail (not necessarily closed) containing every edge of G .

2. **Lemma:** If every vertex of a graph G has degree at least 2, then G contains a cycle.
pf.

- If G contains any loops or multiple edges, the result is trivial. (Assume simple)
- Construct a walk $v_0 \rightarrow v_1 \rightarrow \dots$ by inductively picking v_{i+1} to be any vertex adjacent to v_i . (Hypothesis guarantees existence)
- Since G has only finitely many vertices, we must eventually get a first repeat, v_k .
- Part of walk between occurring between v_k is a cycle.

3. **Theorem** (Euler 1736): A connected graph G is Eulerian if and only if the degree of each vertex of G is even.

pf. \Rightarrow

- Let T be an Eulerian trail of G .
- Whenever T passes through a vertex v , there is a contribution of 2 to $\text{degree}(v)$.
- Since each edge occurs exactly once in T , each vertex must have even degree.

\Leftarrow

- Induction on $|E(G)|$.
- Assume G is connected and degree of each vertex is even.
- By previous lemma, G contains a cycle C . (If C is Eulerian, we're done.)
- If C isn't Eulerian, remove edges of C from G . (Result may be disconnected)
- By induction hypothesis, every component of $H = G - C$ has an Eulerian trail.
- Construct Eulerian trail of G by moving along C from component to component of H always traversing the Eulerian path of a component before moving onto the next.

4. **Corollary:** A connected graph is Eulerian if and only if its set of edges can be split up into disjoint cycles.

5. **Corollary:** A connected graph is semi-Eulerian if and only if it has exactly two vertices of odd degree. (Recall Handshaking Lemma)

6. **Theorem:** (Fleury's algorithm)

Let G be an Eulerian graph. Then the following construction is always possible and produces an Eulerian trail of G .

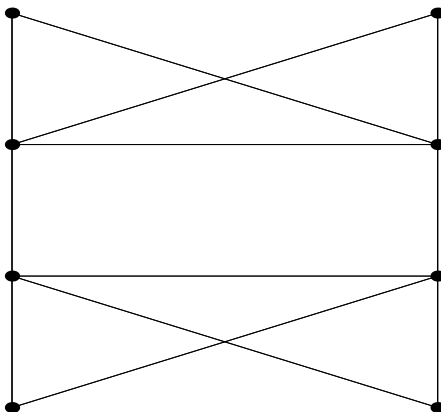
Start at any vertex u and traverse the edges in an arbitrary manner following these rules:

- (i) erase the edges as they are traversed, and erase any isolated vertices that result;
- (ii) at each stage, use a bridge only if there is no alternative.

Proof in book.

7. Example: (p. 34, pr. 6.5)

Use Fleury's algorithm to produce an Eulerian trail for the graph in the following figure.



7 Hamiltonian Graphs

1. Definitions:

- **Hamiltonian cycle**-a cycle passing through every vertex of G exactly once.
- **Hamiltonian graph**-a graph that contains a Hamiltonian cycle.
- **Semi-Hamiltonian graph**-a non-Hamiltonian graph G that contains a path passing through every vertex.
- Sir William Hamilton investigated their existence in the dodecahedron graph.

2. Example: (p. 27, pr. 7.1)

Which of the following graphs are Hamiltonian? semi-Hamiltonian.

(i) K_5

(ii) $K_{2,3}$

(iii) the graph of the octahedron

(iv) the wheel W_6

(v) the 4-cube Q_4

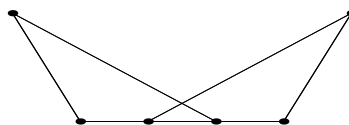
3. **Theorem** (Ore 1960) If G is a simple graph with $n \geq 3$ vertices, and if

$$\deg(v) + \deg(w) \geq n$$

for each pair of non-adjacent vertices v and w , then G is Hamiltonian.

pf.

- Proof by contradiction: assume G has n vertices, satisfies hypotheses, not Hamiltonian.
- We may assume G is almost Hamiltonian: adding any edge will make it Hamiltonian. (Could add extra edges if necessary.)
- G contains a path $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_n$, passing through every vertex, with v_1 not adjacent to v_n (Since G is not Hamiltonian).
- $\deg(v_1) + \deg(v_n) \geq n$.
- There exists (explain) a vertex v_i such that v_1 is adjacent to v_i and v_{i-1} is adjacent to v_n .



- Contradiction, since $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_{i-1} \rightarrow v_n \rightarrow v_{n-1} \rightarrow \dots \rightarrow v_i \rightarrow v_1$ is a Hamiltonian cycle.

4. **Corollary** (Dirac 1952) If G is a simple graph with $n \geq 3$ vertices, and if $\deg(v) > n/2$ for each vertex v , then G is Hamiltonian.

5. Example (p. 37, pr. 7.5)

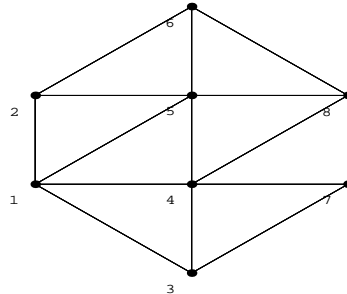
- Prove that, if G is a bipartite graph with an odd number of vertices, then G is non-Hamiltonian.
- Deduce that the graph in Fig. 7.7 (p. 37) is non-Hamiltonian.

8 Some Algorithms

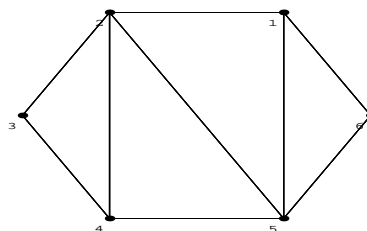
1. **Algorithm** - a finite step-by-step procedure

An algorithm is **efficient** if it provides a “quick” solution

2. The Shortest Path Problem: Given a ‘map’ with road lengths marked, what is the shortest path from one point to another. (p. 42, pr. 8.4)



- **Weighted graph** - a non-negative integer (**weight**) is assigned to each edge e ($w(e)$)
 - Problem: Find a path between two vertices with minimum total weight
 - If each edge has length 1, problem reduces to finding number of edges in shortest path
 - Shortest Path Algorithm
 - Move across the graph from left to right, associating a number $l(V)$ with each vertex (shortest distance from A).
 - Assign A the permanent label 0.
 - At each stage, look at all vertices adjacent to permanent vertices, and make the vertex with the smallest temporary label permanent
 - Shortest path algorithm is efficient
3. The Chinese Postman Problem (Chinese mathematician Mei-Ku Kwan):
A postman wishes to deliver his letters, covering the least possible total distance and returning to his starting point. (Must cover each road in the route at least once and avoid duplication as much as possible.)



- If the graph is Eulerian, then any Eulerian cycle works.
- We consider special case with exactly 2 vertices of odd degree (semi-Eulerian trail).
 - Find a semi-Eulerian path between the two vertices of odd degree.
 - Find a shortest path between the two vertices of odd degree using previous algorithm.
 - Combining the shortest path and the semi-Eulerian trail, we get an Eulerian graph.
- An efficient solution is known in the general case.

4. The Travelling Salesman Problem - A travelling salesman wishes to visit several cities and return to his starting point, covering the least possible distance.

- We want a Hamiltonian cycle of least possible total weight in a weighted complete graph.
- Note that edge weights can also refer to travel times or travel costs.
- If there are 15 cities, the number of Hamiltonian cycles is $(15!)/2 \approx 6.53 \times 10^{11}$.
(Looking at all possible Hamiltonian cycles is not efficient.)
- No efficient algorithm is known.
- There are several heuristic algorithms that quickly tell us *approximately* what the shortest distance is.
- Solve the following travelling salesman problem.

