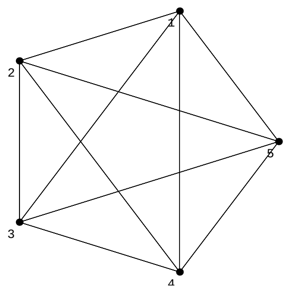
Chapter 5: Planarity

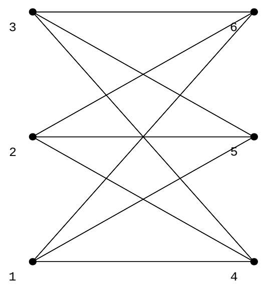
12 Planar Graphs

1. Definitions:
   - **Planar graph** - a graph that can be drawn in the plane without crossings
   - **Plane drawing** or **Plane graph** - a drawing of a graph for which two edges only intersect at a mutually incident vertex

2. Graphs of convex polytopes are planar.

3. **Theorem:** $K_{3,3}$ and K_5 are non-planar.

   pf. $K_{3,3}$

   - $K_{3,3}$ has a 6-cycle $1 \to 4 \to 2 \to 5 \to 3 \to 6 \to 1$. (Any plane drawing must have a hexagon.)
   - The edge $\{1, 5\}$ must line inside or outside the hexagon. (Assume inside–outside similar)
   - Since $\{2, 6\}$ can't cross $\{1, 5\}$, it must lie outside.
   - We can't draw $\{3, 4\}$ as is would cross either $\{2, 6\}$ or $\{1, 5\}$.

   K_5

   - K_5 has a 5-cycle $1 \to 2 \to 3 \to 4 \to 5$. (Any plane drawing must have a pentagon.)
   - Again, assume $\{2, 5\}$ is inside, and $\{1, 3\}$ and $\{1, 4\}$ are outside. (Similar if switched)
   - This implies that $\{2, 4\}$ and $\{3, 5\}$ must both line inside (otherwise would cross $\{1, 3\}$, $\{1, 4\}$ respectively). **Contradiction!**

4. Two graphs are **homeomorphic** if both can be obtained from the same graph by inserting new vertices of degree 2 into its edges.

5. **Theorem** (Kuratowski, 1930): A graph is planar $\Leftrightarrow$ it contains no subgraph homeomorphic to K_5 or $K_{3,3}$.
   (Long proof.)

6. A graph H is **contractible** to K_5 or $K_{3,3}$ if we can obtain K_5 or $K_{3,3}$ by successively contracting edges of H.

7. **Theorem:** A graph is planar $\Leftrightarrow$ it contains no subgraph contractible to K_5 or $K_{3,3}$.
   (Sketch of proof in book.)

8. **Crossing number** (cr(G)): minimum number of crossings that can occur in a plane drawing of G.

9. Example: (p. 64, pr. 12.6)
   Prove that the Petersen graph is non-planar using each of the previous two theorems.
   (Hint: For first theorem, remove the two 'horizontal' edges.)

14 Graphs on Other Surfaces

1. Definitions:

   - **Torus:** doughnut or bagel; can be thought of as a sphere with one handle
   - A surface has **genus** g if it is topologically homeomorphic to a sphere with g handles
   - A graph that can be drawn without crossings on a surface of genus g but not on a surface of genus $g-1$ is a **graph of genus** g**.**

2. **Theorem:** (Proof in book.)
   The genus of a graph does not exceed the crossing number.

3. Example: (p. 72, pr. 14.1)
   The surface of a torus can be regarded as a rectangle in which opposite edges are identified. Using this representation, find drawings of K_5 and $K_{3,3}$ on the torus.

15 Dual Graphs

1. G^* (**geometric dual** of a plane drawing of G):

   (i) Pick a point v^* inside each face of G. These are the vertices of G^*.

   (ii) Any edge e of G divides two faces of G and hence two vertices of G^*. Let e^* be the edge of G^* corresponding to the (dotted) line crossing e.

   Notes:

   - An end vertex or bridge of G corresponds to a loop of G^*.
   - If two faces of G are divided by more than one edge, then G^* has parallel edges.

2. Example: (p. 74, pr. 15.2)
   Show that the dual of the cube graph is the octahedron graph, and that the dual of the dodecahedron graph is the icosahedron graph.

3. **Lemma:** Given a graph G with f-vector (n, m, f) and its geometric dual G^*, with f-vector (n^*, m^*, f^*), we have $n^* = f$; $m^* = m$, $f^* = n$.
   pf. First two are clear from the construction of G^*. Last follows from Euler's formula.

4. **Theorem:** If G is a plane connected graph, then G^{**} is isomorphic to G.
   pf.

   - Process forming G^* from G can be reversed to form G from G^*.
   - We must only check that a face in G^* cannot contain more than one vertex of G.
   - This follows since each face of G^* contains at least one vertex of G, and $n^{**} = f^* = n$.

5. We want a definition of duality that generalizes geometric dual and enables us to determine whether a given graph is planar. (Given a planar graph G, two plane drawings of G may have distinct geometric duals.)

6. **Theorem:** Let G be a planar graph, and let G^* be a geometric dual of G. Then a set of edges in G forms a cycle in G if and only if the corresponding edges in G^* form a cutset.
   pf.

   - $\Rightarrow$ Assume G is a connected plane graph.
   - If C is a cycle of G, then C encloses one or more finite faces of G which correspond to a non-empty set S of vertices of G^*.
   - The edges of G^* crossing C form a cutset of G^*, which disconnect G^* into two subgraphs, with vertex sets S and S-complement, respectively.
   - $\Leftarrow$ Assume C is a cutset of G $\Rightarrow$ removal of edges in C disconnects G into two components S_1 and S_2.
   - Edges in C connect a vertex in S_1 to a vertex in S_2.
   - Edges in C^* form a cycle. (Sketch)

7. **Corollary:** A set of edges of G forms a cutset if and and only if the corresponding set of edges of G^* forms a cycle.

8. G^* is an **abstract dual** of G if there is a one-to-one correspondence between the edges of G and G^*, for which a set of edges of G forms a cycle if and only if the corresponding edges in G^* forms a cutset.

9. **Theorem:** G^* is an abstract dual of $G \Leftrightarrow G$ is an abstract dual of G^*.
   Proof in book.

10. **Theorem:** A graph is planar $\Leftrightarrow$ it has an abstract dual.
    Sketch of proof: ($\Rightarrow$ geometric dual is an abstract dual.)

    - $\Leftarrow$ Assume that G has an abstract dual G^*.
    - Any subgraph of G also has an abstract dual, as removing an edge of G corresponds to contracting an edge of G^*.
    - Inserting or removing a vertex of degree 2 in G results in the addition or deletion of a multiple edge of G^*. It follows that if G' is homeomorphic to G, then G' also has an abstract dual.
    - Book shows that $K_{3,3}$ does not have an abstract dual.
    - Show that K_5 does not have an abstract dual.
    - For contradiction, assume that K_5^* is a dual of $K_5 \Rightarrow K_5$ has 10 edges.
    - Since K_5 only has cycles of length 3, 4, and 5, it follows that every vertex of K_5^* has degree at least 3.
    - Since every cutset of K_5 has either 4 (isolated vertex) or 6 (edge and triangle) edges, every cycle of K_5^* must have either 4 or 6 edges.
    - Let C be a cycle of K_5^* of length 6. If C contains all vertices of K_5^*, then since K_5^* has no triangles, it follows that K_5^* has at most 9 edges, which is a contradiction.
    - If, however, K_5^* contains at least 7 vertices, then it also contains at least $\frac{7 \cdot 3}{2} = 10.5$ edges, which is a contradiction.
    - It follows that K_5 does not have a dual.
    - If G is a non-planar graph, then G has a subgraph homeomorphic to either K_5 or $K_{3,3}$. This implies that $K_{3,3}$ or K_5 has an abstract dual, which is a contradiction.

11. Example: (p. 74, pr. 15.4)
    Use duality to prove that there exists no plane graph with five faces, each pair of which share an edge in common.