Chapter 6: Colouring Graphs

17 Coloring Vertices

1. Sections 17-19: Qualitative (Can we color a graph with given colors?)
   Section 20: Quantitative (How many ways can the coloring be done?)

2. Definitions:

   - If G is a graph without loops, then G is **k-colorable** if we can assign one of k colors to each vertex in such a way that no two adjacent vertices have the same color.
   - If G is k-colorable but not $(k-1)$-colorable, then the **chromatic number** $(\chi(G))$ of G is k (G is **k-chromatic**).

3. Examples:

   - No upper bound on chromatic number of a general graph: $\chi(K_n) = n$.
   - $\chi(G) = 1$ if and only if G is a null graph.
   - $\chi(G) = 2$ if and only if G is a non-null bipartite graph.
   - Examples where $\chi(G) = 3$: C_n and P_n, where n is odd, Petersen graph
   - Examples where $\chi(G) = 4$: W_n, where n is even

4. **Theorem** If G is a simple graph with largest vertex degree Δ, then G is $(\Delta + 1)$-colorable.
   pf.

   - Induction on $|V(G)| = n$.
   - Let G be a simple graph with n vertices.
   - Delete any vertex v: $G' := G - v$ has $n - 1$ vertices and largest vertex degree at most Δ.
   - By IH, G' is $(\Delta + 1)$ colorable.
   - The vertices adjacent to v have at most Δ colors. Color v with any other color.

5. **Theorem**: (Brooks' Theorem, 1941)
   If G is a simple connected graph which is not a complete graph, and if the largest vertex-degree of G is $\Delta(\geq 3)$, then G is Δ-colorable. (Proof in Section 18)

6. Both previous theorems are useful if vertex degrees are approximately the same.
   They tell little if the graph has a few vertices of large degree. ($i.e. K_{1,n}$)

7. **Theorem** Every simple planar graph is 6-colorable.
   pf.

   - Induction on $|V(G)| = n$. (Trivial for $n \leq 6$)
   - Let G be a simple, planar, with n vertices, and assume that all simple planar graphs with at most $n - 1$ vertices are 6-colorable.
   - By Theorem 13.6 (p. 68), G contains a vertex, v, of degree at most 5.
   - $G' := G - v$ is thus 6-colorable.
   - Color G with coloring of G' and coloring v with a color different from the (at most 5) adjacent vertices.

8. **Theorem**: Every simple planar graph is 5-colorable.
   pf.

   - Induction on $|V(G)| = n$. (Trivial for $n < 6$)
   - Let G be a simple, planar, with n vertices, and assume that all simple planar graphs with at most $n-1$ vertices are 5-colorable.
   - By Theorem 13.6 (p.68), G contains a vertex, v, of degree at most 5.
   - $G' := G - v$ is thus 5-colorable.
   - If $\deg(v) < 5$, then v can be colored by any color not adjacent to v.
   - Assume $\deg(v) = 5$, adjacent to vertices $v_1, \ldots, v_5$.
   - If $v_1, \ldots, v_5$ are mutually adjacent, then G contains K_5 as a subgraph, which is impossible since G is planar.
   - Hence at least 2 vertices (WLOG v_1 and v_2) are not adjacent.
   - Contract edges vv_1 and vv_2. Result is planar with fewer than n vertices $\Rightarrow$ 5-colorable.
   - Now color v_1 and v_2 with the color originally assigned to v (w/ edges contracted).
   - A 5-coloring of G is obtained by color v differently than the (at most 4) colors assigned to $v_1, \ldots, v_5$.

9. **Theorem**: (Appel and Haken, 1976)
   Every simple planar graph is 4-colorable.

10. Example: (p. 86, pr. 17.7)
    Let G be a simple graph with n vertices, which is regular of degree d. By considering the number of vertices that can be assigned the came color, prove that $\chi(G) \geq n/(n-d)$.

11. Example: (p. 86, pr. 17.8) Let G be a simple planar graph containing no triangles.

    (a) Using Euler's formula, show that G contains a vertex of degree at most 3.

    (b) Use induction to show that G is 4-colorable.

    (In fact, it can be proved that G is 3-colorable.)

19 Colouring Maps

1. *The 4-color problem*

   - Whether a map can be colored with 4 colors so that no 2 adjacent countries are shown in the same color.
   - Raised by Francis Gurthrie in 1852.
   - Presented to the general public (London Mathematical Society) by Cayley in 1878.
   - Kempe published an incorrect proof in 1879, modified by Heawood in 1890 into a proof of the five color theorem.
   - First generally accepted proof by Appel and Haken in 1977 (builds on Kempe's ideas).
     - First shows that every plane triangulation must contain at least one of 1,482 'unavoidable configurations.'
     - Second, a computer is used to show that each configuration is 'reducible', meaning that any plane triangulation containing such a configuration can be 4-colored by piecing together 4-colorings of smaller plane triangulations.
     - Together, these produce an inductive proof that all plane triangulations, and hence all planar graphs can be 4-colored.

     Proof criticized, responded with 741 page long algorithmic version of their proof.
   - Shorter proof more readily verifiable given by N. Robertson, D. Sanders, P.D. Seymour, and R. Thomas in 1997.

2. Definitions:

   - A **map** is a 3-connected plane graph; it contains no cutsets with 1 or 2 edges, no vertices of degree 1 or 2.
   - A map is **k-colorable(f)** if its faces can be colored with k colors with no adjacent faces having the same color.
   - A graph is **k-colorable(v)** if its k-colorable, as in Section 17.

3. **Theorem:** Let G be a plane graph without loops, and let G^* be geometric dual of G. Then G is k-colorable(v) $\Leftrightarrow$ G^* is k-colorable(f).
   pf. $\Rightarrow$

   - Assume G is simple and connected $\Rightarrow$ G^* is a map.
   - Assuming we have a k-coloring(v) of G, color each face of G^* with the color of the corresponding vertex in G.
   - No two adjacent faces of G^* have the same color because the vertices they correspond to in G are adjacent and have different colors.
   - Thus, G^* is k-colorable(f).

   $\Leftarrow$

   - Suppose we have a k-coloring(f) of G^*.
   - k-color the vertices of G so that each vertex has the color of the face in G^* containing it.
   - Again, no two adjacent vertices of G have the same color, and G is k-colorable.

4. Example: (p. 92, pr. 19.3)
   Give an example of a plane graph that is both 2-colorable(f) and 2-colorable(v).

5. **Theorem:** A map G is 2-colorable(f) $\Leftrightarrow$ G is an Eulerian graph.
   pf. $\Rightarrow$ For every $v \in V(G)$, even # of faces at v since covered with 2 colors.
   G is Eulerian since every vertex hence has even degree.
   Alternate Proof in whole:

   - By Exercise 15.9, G is Eulerian $\Leftrightarrow$ G^* is bipartite.
   - A connected graph without loops hence is 2-colorable $\Leftrightarrow$ it is bipartite.

6. **Corollary:** The four-color theorem for maps is equivalent to the four-color theorem for planar graphs.
   Proof in book.

7. **Theorem:** Let G be a cubic map. Then G is 3-colorable(f) $\Leftrightarrow$ each face is bounded by an even number of edges.
   pf. $\Rightarrow$

   - Given any face F, the faces surrounding F must alternate in color.
   - There must be an even number of them, so each face is bounded by even # of edges.

   $\Leftarrow$

   - We prove the dual result.
   - Assume G is a simple connected plane graph where each face is a triangle and every vertex has even degree ($\Rightarrow$ G is Eulerian).
   - We must prove that G is 3-colorable(v) with colors r, y, g.
   - By Theorem, G Eulerian $\Rightarrow$ G is 2-colorable(f), with colors black and whit.
   - Color any white face so that r, y, g appear in clockwise order, counter-clockwise black faces.
   - Vertex coloring is extended to the whole graph.

8. **Theorem:** In order to prove the four-color theorem, it is sufficient to prove that each cubic map is 4-colorable(f).
   pf.

   - Corollary above implies enough to show that 4-colorability(f) of every cubic map $\Rightarrow$ 4-colorability(f) of any map.
   - Let G be a map, and assume that every cubic map is 4-colorable(f).
   - Remove vertices of degree 2 without affecting coloring.
   - Only remains to eliminate vertices of degree ≥ 4.
   - If v has degree $n \geq 4$, then cover v with an n-gon patch.
   - Repeating this process for all such vertices, we obtain a cubic map that's 4-colorable(f) by hypothesis.
   - 4-coloring of faces of G is obtained by shrinking each patch to a single vertex and reinstating each patch of degree 2.

20 Colouring Edges

1. Definitions:

   - G is **k-colorable(e)** (or k-edge colorable) if its edges can be colored with k colors so that no two adjacent edges have the same color.
   - The **chromatic index**, $\chi'(G)$ is the number k such that G is k-colorable(e) but not $(k-1)$-colorable(e).

2. **Theorem** (Vizing, 1964)
   If G is a simple graph with largest vertex-degree Δ, then

$$\Delta \leq \chi'(G) \leq \Delta + 1.$$

3. Chromatic index for particular graphs:

   - $\chi'(C_{2n}) = 2$, and $\chi'(C_{2n+1}) = 3$.
   - $\chi'(W_n) = n - 1$, if $n \geq 4$.
   - Example: (p. 95, pr. 20.5) Chromatic index of Platonic graphs?

4. **Theorem:** $\chi'(K_n) = n$ if n is odd ($n \neq 1$), and $\chi'(K_n) = n - 1$ if n is even.
   pf.

   - Assume $n \geq 3$. (Otherwise trivial)
   - If n is odd, place the vertices as a regular n-gon.
   - Color n-cycle with a different color for each edge.
   - Color remaining edges with color of boundary edge parallel to it.
   - K_n is not $(n-1)$-colorable(e), as the largest number of edges of the same color is $(n-1)/2$.
   - It follows that K_n has at most $(n-1)/2 \cdot \chi'(K_n) = (n-1)^2/2$ edges.
   - If n is even, K_n can be built by attaching a new vertex to all vertices in K_{n-1}.
   - Color K_{n-1} as before.
   - One color is missing at each vertex, and the colors are all different.
   - Color "new" edges of K_n with the missing colors.

5. **Theorem:** The four-color theorem is equivalent to the statement that $\chi'(G) = 3$ for each cubic map G.
   pf. $\Rightarrow$

   - Since G is cubic, each vertex is surrounded by a tetrahedron.
   - Assume G is 4-colored(f) by $\alpha = (1,0)$, $\beta = (0,1)$, $\gamma = (1,1)$, and $\delta = (0,0)$.
   - Construct a 3-coloring(e) by coloring each edge e by the sum of the colors of the two adjacent faces, mod 2. (Example)
   - Δ cannot occur in edge coloring as 2 faces adjacent to each edge must have different colors.
   - Furthermore, no two adjacent edges can share the same color.

   $\Leftarrow$

   - Suppose we have a 3-coloring(e) for G: α, β, $\gamma \Rightarrow$ edge of each color at each vertex.
   - Subgraph determine by edges of any pair of colors (i.e. α & β) is 2-regular, hence Eulerian.
   - Theorem 19.1 (2-colorable(f) $\Leftrightarrow$ Eulerian) implies that we can color its faces with two colors, 0 and 1.
   - Doing this for each pair of colors, each edge is assigned two colors (x, y), where each is 0 or 1.
   - Since coordinates of two adjacent faces must differ in at least one place, (1,0), (0,1), (1,1), (0,0) give the required 4-coloring(f).

6. **Theorem:** (König 1916)
   If G is a bipartite graph with largest vertex-degree Δ, then $\chi'(G) = \Delta$.
   Proof in Book.

7. **Corollary:** $\chi'(K_{r,s} = \max(r, s)$.

8. Example: (p. 95, pr. 20.7)
   Prove that if G is a cubic Hamiltonian graph, then $\chi'(G) = 3$.

21 Chromatic Polynomials

1. $P_G(k) = \#$ of ways to color the vertices of G with k-colors
   ($P_G(k)$ is the **chromatic function** of G, soon to be **chromatic polynomial**.)

2. Chromatic Functions:

   - Determine $P_G(k)$ for paths, trees, K_n, C_n.
     In each case, determine how many ways they can be colored with 5 or 6 colors.
   - Observe that if $k < \chi(G)$, then $P_G(k) = 0$, and $k \geq \chi(G) \Rightarrow P_G(k) > 0$.
   - Four-color Theorem is equivalent to: If G is a simple planar graph, then $P_G(4) > 0$.

3. **Theorem:** Let G be a simple graph, and let $G - e$ and $G\backslash e$ be the graphs obtained from G by deleting and contracting e. Then

$$P_G(k) = P_{G-e}(k) - P_{G\backslash e}(k).$$

   Give an example.
   pf. (Book includes some G/e.)

   - Equivalent to showing that $P_{G-e}(k) = P_G(k) + P_{G\backslash e}(k)$.
   - Let $e = vw$.
   - $\#$ of k-colorings of $G - e$ in which v and w have **different** colors is unchanged if e is added $\Rightarrow$ equals $P_G(k)$.
   - $\#$ of k-colorings of $G - e$ in which v and w have **the same** color is unchanged if v and w are identified with each other $\Rightarrow$ equals $P_{G\backslash e}(k)$.
   - Thus, $P_{G-e}(k) = P_G(k) + P_{G\backslash e}(k)$.

4. **Corollary:** The chromatic function of a simple graph is a polynomial.
   pf.

   - Induction on the number of edges.
   - Basis: 0 edges $\Rightarrow P_G(k) = k^n$, where n is the number of vertices (components).
   - Assume true for graphs with m or fewer edges, and let G have $m + 1$ edges.
   - $G - e$ and $G\backslash e$ have m edges, and hence $P_{G-e}(k)$ and $P_{G\backslash e}(k)$ are polynomials.
   - Result follows since $P_G(k) = P_{G-e}(k) - P_{G\backslash e}(k)$.

   Note: Quickly follows that G has n vertices $\Rightarrow P_G(k)$ is of degree n, and the coefficient of k^n is 1.

5. Example: (p. 99, pr. 21.4)

   (a) Prove that the chromatic polynomial of $K_{2,s}$ is

$$k(k-1)^s + k(k-1)(k-2)^s.$$

   (b) Prove that the chromatic polynomial of C_n is

$$(k-1)^n + (-1)^n(k-1).$$