Research Statement

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Research Background

My research interests include, but are not limited to, discrete mathematics or combinatorics. To this point, my research has more specifically dealt with polytopes. A polytope P is a bounded polyhedron, and can be defined as the convex hull of a finite point set, or equivalently as the bounded intersection of finitely many closed halfspaces. The dimension, d, of P is one less than the maximum number of affinely independent points contained in it, and a polytope of dimension d is often referred to as a d-polytope. A face of P is the intersection of a supporting hyperplane of P with P itself. Any face of a polytope is itself a polytope, and the dimension of a face is its dimension as a polytope. The faces of dimension 0, 1, $d-2$, and $d-1$ are often referred to as vertices, edges, subfacets, and facets, respectively. The d-simplex is the d-polytope with the minimum number of vertices, $d+1$. This gives rise to a class of polytopes, known as simplicial polytopes, whose facets and consequently all proper faces are simplices.

The f-vector of a d-polytope P lists the number of faces of each dimension from 0 up to $d-1$. The flag f-vector contains the f-vector, but it also enumerates the numbers of chains of faces of P. Bayer and Klapper [1] proved that the data in the flag f-vector can be equivalently stored in the cd-index, which removes the redundant information present in the flag f-vector. A major combinatorial problem is to characterize the f-vectors and flag f-vectors (or equivalently cd-indices) of d-polytopes. For 3-polytopes, the problem was solved by Steinitz nearly a century ago. It was also solved for the class of simplicial polytopes by Stanley [2] and Billera and Lee [3] more than twenty years ago. For $d \geq 4$, however, the problem of characterizing the flag f-vectors of general d-polytopes is open. Although knowledge of 4-polytopes is constantly increasing, [4] and [5] give overviews of what is currently known in the case of $d = 4$.

Grünbaum [6] provided a tool for determining the facial structure of a polytope obtained by adding a single point to a known polytope. Using this, Shemer [7] developed a process called sewing, which he used to inductively create a large class of simplicial polytopes called neighborly polytopes. Bisztriczky [8], [9] developed another class of polytopes, ordinary polytopes, that are a generalization of neighborly cyclic polytopes that contains non-simplicial polytopes as well. Dinh proved that ordinary polytopes are realizable, and I have since provided an alternate proof to Dinh's using a modified version of Shemer's sewing, which I call A-sewing.

Research Results

- Using A-sewing, I created several large class of 4-polytopes, including classes of non-simplicial 4-polytopes.

- I proved that the cd-index of a 4-polytope does not decrease when any sewing or A-sewing operation is performed.

- I determined and proved bounds that the changes in the flag f-vector are subject to with regard to sewing and A-sewing.

Open Research Problems

- Is the cd-index is non-decreasing with respect to sewing and A-sewing for all d-polytopes ($d \geq 5$)?
- Characterize the flag f-vectors (or equivalently cd-indices) of 4-polytopes.
- Are there other operations that when combined with sewing and A-sewing can produce all flag f-vectors of realizable 4-polytopes?

References

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